Chapter 1

SETS

1.1 Overview

This chapter deals with the concept of a set, operations on sets.Concept of sets will be useful in studying the relations and functions.

1.1.1 ***Set and their representations*** A set is a well-defined collection of objects. There are two methods of representing a set

(i) Roaster or tabular form (ii) Set builder form

1.1.2 ***The empty set*** A set which does not contain any element is called the empty set or the void set or null set and is denoted by { } or ϕ.

1.1.3 ***Finite and infinite sets*** A set which consists of a finite number of elements is called a finite set otherwise, the set is called an infinite set.

1.1.4 ***Subsets*** A set A is said to be a subset of set B if every element of A is also an element of B. In symbols we write $A \subset B$ if $a \in A \Rightarrow a \in B$.

We denote set of real numbers by **R**
set of natural numbers by **N**
set of integers by **Z**
set of rational numbers by **Q**
set of irrational numbers by **T**

We observe that

$$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R},$$
$$\mathbf{T} \subset \mathbf{R},\ \mathbf{Q} \not\subset \mathbf{T},\ \mathbf{N} \not\subset \mathbf{T}$$

1.1.5 ***Equal sets*** Given two sets A and B, if every elements of A is also an element of B and if every element of B is also an element of A, then the sets A and B are said to be equal. The two equal sets will have exactly the same elements.

1.1.6 ***Intervals as subsets of R*** Let $a, b \in$ R and $a < b$. Then

(a) An open interval denoted by (a, b) is the set of real numbers $\{x : a < x < b\}$

(b) A closed interval denoted by $[a, b]$ is the set of real numbers $\{x : a \le x \le b)$

(c) Intervals closed at one end and open at the other are given by

$[a, b) = \{x : a \le x < b\}$

$(a, b] = \{x : a < x \le b\}$

1.1.7 *Power set* The collection of all subsets of a set A is called the **power set** of A. It is denoted by P(A). If the number of elements in A = n , i.e., n(A) = n, then the number of elements in P(A) = 2^n.

1.1.8 *Universal set* This is a basic set; in a particular context whose elements and subsets are relevant to that particular context. For example, for the set of vowels in English alphabet, the universal set can be the set of all alphabets in English. Universal set is denoted by **U**.

1.1.9 *Venn diagrams* Venn Diagrams are the diagrams which represent the relationship between sets. For example, the set of natural numbers is a subset of set of whole numbers which is a subset of integers. We can represent this relationship through Venn diagram in the following way.

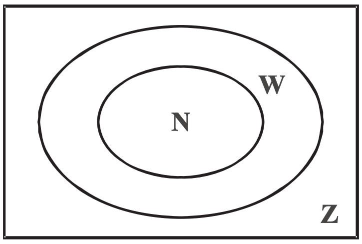


Fig 1.1

1.1.10 *Operations on sets*

Union of Sets : The union of any two given sets A and B is the set C which consists of all those elements which are either in A or in B. In symbols, we write

$$C = A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

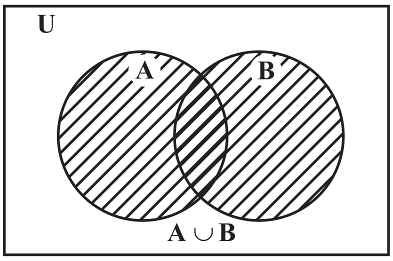


Fig 1.2 (a)

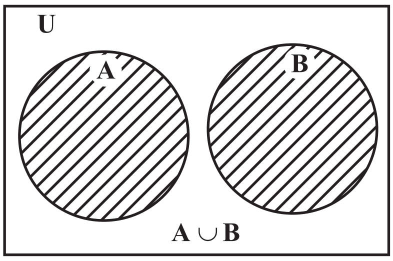


Fig 1.2 (b)

Some properties of the operation of union.

(i) $A \cup B = B \cup A$

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$

(iii) $A \cup \phi = A$

(iv) $A \cup A = A$

(v) $U \cup A = U$

Intersection of sets: The intersection of two sets A and B is the set which consists of all those elements which belong to both A and B. Symbolically, we write $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

When $A \cap B = \phi$, then A and B are called disjoint sets.

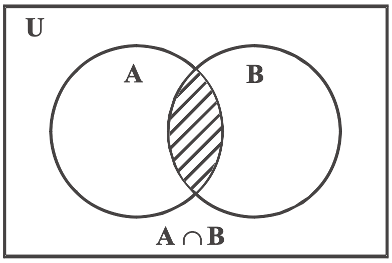


Fig 1.3 (a)

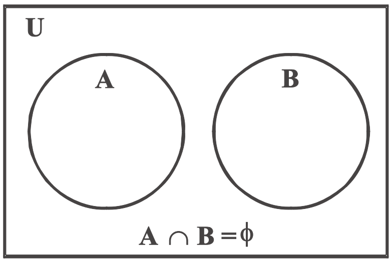


Fig 1.3 (b)

Some properties of the operation of intersection

(i) $A \cap B = B \cap A$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

(iii) $\phi \cap A = \phi$; $U \cap A = A$

(iv) $A \cap A = A$

(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(vi) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Difference of sets The difference of two sets A and B, denoted by A – B is defined as set of elements which belong to A but not to B. We write

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

also,

$$B - A = \{ x : x \in B \text{ and } x \notin A\}$$

Complement of a set Let U be the universal set and A a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. Symbolically, we write

$$A' = \{x : x \in U \text{ and } x \notin A\}. \text{ Also } A' = U - A$$

Some properties of complement of sets

(i) Law of complements:

(a) $A \cup A' = U$

(b) $A \cap A' = \phi$

(ii) De Morgan's law

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

(iii) $(A')' = A$

(iv) $U' = \phi$ and $\phi' = U$

1.1.11 *Formulae to solve practical problems on union and intersection of two sets*

Let A, B and C be any finite sets. Then

(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(b) If $(A \cap B) = \phi$, then $n(A \cup B) = n(A) + n(B)$

(c) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

1.2 Solved Examples

Short Answer Type

Example 1 Write the following sets in the roaster form.

(i) $A = \{x \mid x$ is a positive integer less than 10 and $2^x - 1$ is an odd number$\}$

(ii) $C = \{x : x^2 + 7x - 8 = 0, x \in \mathbf{R}\}$

Solution

(i) $2^x - 1$ is always an odd number for all positive integral values of x. In particular, $2^x - 1$ is an odd number for $x = 1, 2, ..., 9$. Thus, $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(ii) $x^2 + 7x - 8 = 0$ or $(x + 8)(x - 1) = 0$ giving $x = -8$ or $x = 1$

Thus, $C = \{-8, 1\}$

Example 2 State which of the following statements are true and which are false. Justify your answer.

(i) $37 \notin \{x \mid x$ has exactly two positive factors$\}$

(ii) $28 \in \{y \mid$ the sum of the all positive factors of y is $2y\}$

(iii) $7,747 \in \{t \mid t$ is a multiple of $37\}$

Solution

(i) False

Since, 37 has exactly two positive factors, 1 and 37, 37 belongs to the set.

(ii) True

Since, the sum of positive factors of 28

$$= 1 + 2 + 4 + 7 + 14 + 28$$
$$= 56 = 2(28)$$

(iii) False

7,747 is not a multiple of 37.

Example 3 If X and Y are subsets of the universal set U, then show that

(i) $Y \subset X \cup Y$ (ii) $X \cap Y \subset X$ (iii) $X \subset Y \Rightarrow X \cap Y = X$

Solution

(i) $X \cup Y = \{x \mid x \in X$ or $x \in Y\}$

Thus $x \in Y \Rightarrow x \in X \cup Y$

Hence, $Y \subset X \cup Y$

(ii) $X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$

Thus $x \in X \cap Y \Rightarrow x \in X$

Hence $X \cap Y \subset X$

(iii) Note that

$x \in X \cap Y \Rightarrow x \in X$

Thus $X \cap Y \subset X$

Also, since $X \subset Y$,

$x \in X \Rightarrow x \in Y \Rightarrow x \in X \cap Y$

so that $X \subset X \cap Y$

Hence the result $X = X \cap Y$ follows.

Example 4 Given that $N = \{1, 2, 3, ..., 100\}$, then

(i) Write the subset A of N, whose element are odd numbers.

(ii) Write the subset B of N, whose element are represented by $x + 2$, where $x \in N$.

Solution

(i) $A = \{x \mid x \in N \text{ and } x \text{ is odd}\} = \{1, 3, 5, 7, ..., 99\}$

(ii) $B = \{y \mid y = x + 2, x \in N\}$

So, for $1 \in N$, $y = 1 + 2 = 3$

$2 \in N$, $y = 2 + 2 = 4$,

and so on. Therefore, $B = \{3, 4, 5, 6, ... , 100\}$

Example 5 Given that $E = \{2, 4, 6, 8, 10\}$. If n represents any member of E, then, write the following sets containing all numbers represented by

(i) $n + 1$ (ii) n^2

Solution Given $E = \{2, 4, 6, 8, 10\}$

(i) Let $A = \{x \mid x = n + 1, n \in E\}$

Thus, for $2 \in E$, $x = 3$

$4 \in E$, $x = 5$,

and so on. Therefore, $A = \{3, 5, 7, 9, 11\}$.

(ii) Let $B = \{x \mid x = n^2, n \in E\}$

So, for $2 \in E$, $x = (2)^2 = 4$, $4 \in E$, $x = (4)^2 = 16$, $6 \in E$, $x = (6)^2 = 36$,

and so on. Hence, $B = \{4, 16, 36, 64, 100\}$

Example 6 Let $X = \{1, 2, 3, 4, 5, 6\}$. If n represent any member of X, express the following as sets:

(i) $n \in$ X but $2n \notin$ X (ii) $n + 5 = 8$

(iii) n is greater than 4.

Solution

(i) For X = {1, 2, 3, 4, 5, 6}, it is the given that $n \in$ X, but $2n \notin$ X.

Let, A = $\{x \mid x \in X \text{ and } 2x \notin X\}$

Now, $1 \notin A$ as $2.1 = 2 \in X$

$2 \notin A$ as $2.2 = 4 \in X$

$3 \notin A$ as $2.3 = 6 \in X$

But $4 \in A$ as $2.4 = 8 \notin X$

$5 \in A$ as $2.5 = 10 \notin X$

$6 \in A$ as $2.6 = 12 \notin X$

So, A = {4, 5, 6}

(ii) Let B = $\{x \mid x \in X \text{ and } x + 5 = 8\}$

Here, B = {3}

as $x = 3 \in X$ and $3 + 5 = 8$ and there is no other element belonging to X such that $x + 5 = 8$.

(iii) Let C = $\{x \mid x \in X, x > 4\}$

Therefore, C = {5, 6}

Example 7 Draw the Venn diagrams to illustrate the followoing relationship among sets E, M and U, where E is the set of students studying English in a school, M is the set of students studying Mathematics in the same school, U is the set of all students in that school.

(i) All the students who study Mathematics study English, but some students who study English do not study Mathematics.

(ii) There is no student who studies both Mathematics and English.

(iii) Some of the students study Mathematics but do not study English, some study English but do not study Mathematics, and some study both.

(iv) Not all students study Mathematics, but every students studying English studies Mathematics.

Solution

(i) Since all of the students who study mathematics study English, but some students who study English do not study Mathematics.

Therefore, $M \subset E \subset U$

Thus the Venn Diagram is

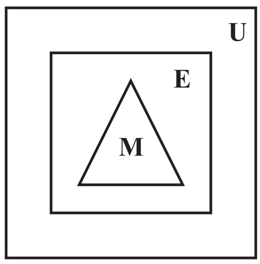


Fig 1.4

(ii) Since there is no student who study both English and Mathematics

Hence, $E \cap M = \phi$.

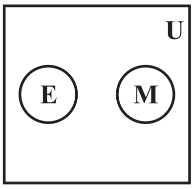


Fig 1.5

(iii) Since there are some students who study both English and Mathematics, some English only and some Mathematics only.

Thus, the Venn Diagram is

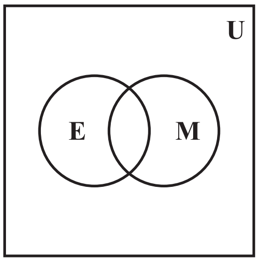


Fig 1.6

(iv) Since every student studying English studiesMathematics.

Hence, $E \subset M \subset U$

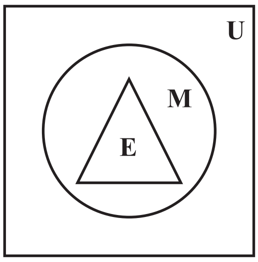


Fig 1.7

Example 8 For all sets A, B and C

Is $(A \cap B) \cup C = A \cap (B \cup C)$?

Justify your statement.

Solution No. consider the following sets A, B and C :

$$A = \{1, 2, 3\}$$
$$B = \{2, 3, 5\}$$
$$C = \{4, 5, 6\}$$

Now $(A \cap B) \cup C = (\{1, 2, 3\} \cap \{2, 3, 5\}) \cup \{4, 5, 6\}$

$= \{2, 3\} \cup \{4, 5, 6\}$

$= \{2, 3, 4, 5, 6\}$

And $A \cap (B \cup C) = \{1, 2, 3\} \cap [\{2, 3, 5\} \cup \{4, 5, 6\}$

$= \{1, 2, 3\} \cap \{2, 3, 4, 5, 6\}$

$= \{2, 3\}$

Therefore, $(A \cap B) \cup C \neq A \cap (B \cup C)$

Example 9 Use the properties of sets to prove that for all the sets A and B

$$A - (A \cap B) = A - B$$

Solution We have

$A - (A \cap B) = A \cap (A \cap B)'$ (since $A - B = A \cap B'$)

$= A \cap (A' \cup B')$ [by De Morgan's law)

$= (A \cap A') \cup (A \cap B')$ [by distributive law]

$= \phi \cup (A \cap B')$

$= A \cap B' = A - B$

Long Answer Type

Example 10 For all sets A, B and C

Is $(A - B) \cap (C - B) = (A \cap C) - B$?

Justify your answer.

Solution Yes

Let $x \in (A - B) \cap (C - B)$

$\Rightarrow$ $x \in A - B$ and $x \in C - B$

$\Rightarrow$ $(x \in A$ and $x \notin B)$ and $(x \in C$ and $x \notin B)$

$\Rightarrow$ $(x \in A$ and $x \in C)$ and $x \notin B$

$\Rightarrow$ $(x \in A \cap C)$ and $x \notin B$

$\Rightarrow$ $x \in (A \cap C) - B$

So $(A - B) \cap (C - B) \subset (A \cap C) - B$ ... (1)

Now, conversely

Let $\quad y \in (A \cap C) - B$

$\Rightarrow \quad y \in (A \cap C)$ and $y \notin B$

$\Rightarrow \quad (y \in A$ and $y \in C)$ and $(y \notin B)$

$\Rightarrow \quad (y \in A$ and $y \notin B)$ and $(y \in C$ and $y \notin B)$

$\Rightarrow \quad y \in (A - B)$ and $y \in (C - B)$

$\Rightarrow \quad y \in (A - B) \cap (C - B)$

So $\quad (A \cap C) - B \subset (A - B) \cap (C - B)$ ... (2)

From (1) and (2), $(A - B) \cap (C - B) = (A \cap C) - B$

Example 11 Let A, B and C be sets. Then show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution We first show that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C)$. Then

$\quad x \in A \quad$ or $\quad x \in B \cap C$

$\Rightarrow \quad x \in A \quad$ or $\quad (x \in B$ and $x \in C)$

$\Rightarrow \quad (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow \quad (x \in A \cup B)$ and $\quad (x \in A \cup C)$

$\Rightarrow \quad x \in (A \cup B) \cap (A \cup C)$

Thus, $\quad A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ ... (1)

Now we will show that $(A \cup B) \cap (A \cup C) \subset (A \cup C)$

Let $\quad x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow \quad x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow \quad (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow \quad x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow \quad x \in A$ or $(x \in B \cap C)$

$\Rightarrow \quad x \in A \cup (B \cap C)$

Thus, $\quad (A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ ... (2)

So, from (1) and (2), we have

$$A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$$

Example 12 Let P be the set of prime numbers and let $S = \{t \mid 2^t - 1$ is a prime$\}$. Prove that $S \subset P$.

Solution Now the equivalent contrapositive statement of $x \in S \Rightarrow x \in P$ is $x \notin P \Rightarrow x \notin S$.

Now, we will prove the above contrapositive statement by contradiction method

Let $x \notin P$

$\Rightarrow$ x is a composite number

Let us now assume that $x \in S$

$\Rightarrow$ $2^x - 1 = m$ (where m is a prime number)

$\Rightarrow$ $2^x = m + 1$

Which is not true for all composite number, say for $x = 4$ because

$2^4 = 16$ which can not be equal to the sum of any prime number m and 1.

Thus, we arrive at a contradiction

$\Rightarrow$ $x \notin S$.

Thus, when $x \notin P$, we arrive at $x \notin S$

So $S \subset P$.

Example 13 From 50 students taking examinations in Mathematics, Physics and Chemistry, each of the student has passed in at least one of the subject, 37 passed Mathematics, 24 Physics and 43 Chemistry. At most 19 passed Mathematics and Physics, at most 29 Mathematics and Chemistry and at most 20 Physics and Chemistry. What is the largest possible number that could have passed all three examination?

Solution Let M be the set of students passing in Mathematics

P be the set of students passing in Physics

C be the set of students passing in Chemistry

Now, $n(M \cup P \cup C) = 50$, $n(M) = 37$, $n(P) = 24$, $n(C) = 43$

$n(M \cap P) \leq 19$, $n(M \cap C) \leq 29$, $n(P \cap C) \leq 20$ (Given)

$$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C) \leq 50$$

$\Rightarrow$ $37 + 24 + 43 - 19 - 29 - 20 + n(M \cap P \cap C) \leq 50$

$\Rightarrow$ $n(M \cap P \cap C) \leq 50 - 36$

$\Rightarrow$ $n(M \cap P \cap C) \leq 14$

Thus, the largest possible number that could have passed all the three examinations is 14.

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 14 to 16: (M.C.Q.)

Example 14 Each set X_r contains 5 elements and each set Y_r contains 2 elements and $\bigcup_{r=1}^{20} X_r = S = \bigcup_{r=1}^{n} Y_r$. If each element of S belong to exactly 10 of the X_r's and to exactly 4 of the Y_r's, then n is

(A) 10 (B) 20 (C) 100 (D) 50

Solution The correct answer is (B)

Since, $n(X_r) = 5$, $\bigcup_{r=1}^{20} X_r = S$, we get $n(S) = 100$

But each element of S belong to exactly 10 of the X_r's

So, $\frac{100}{10} = 10$ are the number of distinct elements in S.

Also each element of S belong to exactly 4 of the Y_r's and each Y_r contain 2 elements. If S has n number of Y_r in it. Then

$$\frac{2n}{4} = 10$$

which gives $n = 20$

Example 15 Two finite sets have m and n elements respectively. The total number of subsets of first set is 56 more than the total number of subsets of the second set. The values of m and n respectively are.

(A) 7, 6 (B) 5, 1 (C) 6, 3 (D) 8, 7

Solution The correct answer is (C).

Since, let A and B be such sets, i.e., $n(A) = m$, $n(B) = n$

So $n(P(A)) = 2^m$, $n(P(B)) = 2^n$

Thus $n(P(A)) - n(P(B)) = 56$, i.e., $2^m - 2^n = 56$

$\Rightarrow$ $2^n(2^{m-n} - 1) = 2^3 7$

$\Rightarrow$ $n = 3$, $2^{m-n} - 1 = 7$

$\Rightarrow$ $m = 6$

Example 16 The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to

(A) $B \cap C'$ (B) $A \cap C$ (C) $B \cup C'$ (D) $A \cap C'$

Solution The correct choice is (A).

Since $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$

$$= (A \cup (B \cup C)) \cap (A' \cup (B \cup C)) \cap C'$$
$$= (A \cap A') \cup (B \cup C) \cap C'$$
$$= \phi \cup (B \cup C) \cap C'$$
$$= B \cap C' \cup \phi = B \cap C'$$

Fill in the blanks in Examples 17 and 18 :

Example 17 If A and B are two finite sets, then $n(A) + n(B)$ is equal to ___________

Solution Since $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

So $n(A) + n(B) = n(A \cup B) + n(A \cap B)$

Example 18 If A is a finite set containing n element, then number of subsets of A is ___________

Solution 2^n

State true or false for the following statements given in Examples 19 and 20.

Example 19 Let R and S be the sets defined as follows:

$$R = \{x \in \mathbf{Z} \mid x \text{ is divisible by } 2\}$$
$$S = \{y \in \mathbf{Z} \mid y \text{ is divisible by } 3\}$$

then $R \cap S = \phi$

Solution False

Since 6 is divisible by both 3 and 2.

Thus $R \cap S \neq \phi$

Example 20 $\mathbf{Q} \cap \mathbf{R} = \mathbf{Q}$, where $\mathbf{Q}$ is the set of rational numbers and $\mathbf{R}$ is the set of real numbers.

Solution True

Since $\mathbf{Q} \subset \mathbf{R}$

So $\mathbf{Q} \cap \mathbf{R} = \mathbf{Q}$

1.3 EXERCISE

Short Answer Type

1. Write the following sets in the roaster from

   (i) $A = \{x : x \in \mathbf{R}, 2x + 11 = 15\}$ (ii) $B = \{x \mid x^2 = x, x \in \mathbf{R}\}$

   (iii) $C = \{x \mid x$ is a positive factor of a prime number $p\}$

2. Write the following sets in the roaster form :

(i) $D = \{t \mid t^3 = t, t \in R\}$ (ii) $E = \{w \mid \dfrac{w-2}{w+3} = 3, w \in \mathbf{R}\}$

(iii) $F = \{x \mid x^4 - 5x^2 + 6 = 0, x \in \mathbf{R}\}$

3. If $Y = \{x \mid x$ is a positive factor of the number $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a prime number$\}$.Write Y in the roaster form.

4. State which of the following statements are true and which are false. Justify your answer.

(i) $35 \in \{x \mid x$ has exactly four positive factors$\}$.

(ii) $128 \in \{y \mid$ the sum of all the positive factors of y is $2y\}$

(iii) $3 \notin \{x \mid x^4 - 5x^3 + 2x^2 - 112x + 6 = 0\}$

(iv) $496 \notin \{y \mid$ the sum of all the positive factors of y is $2y\}$.

5. Given $L = \{1, 2, 3, 4\}$, $M = \{3, 4, 5, 6\}$ and $N = \{1, 3, 5\}$

Verify that $L - (M \cup N) = (L - M) \cap (L - N)$

6. If A and B are subsets of the universal set U, then show that

(i) $A \subset A \cup B$ (ii) $A \subset B \Leftrightarrow A \cup B = B$

(iii) $(A \cap B) \subset A$

7. Given that $N = \{1, 2, 3, ..., 100\}$. Then write

(i) the subset of N whose elements are even numbers.

(ii) the subset of N whose element are perfect square numbers.

8. If $X = \{1, 2, 3\}$, if n represents any member of X, write the following sets containing all numbers represented by

(i) $4n$ (ii) $n + 6$ (iii) $\dfrac{n}{2}$ (iv) $n - 1$

9. If $Y = \{1, 2, 3, ... 10\}$, and a represents any element of Y, write the following sets, containing all the elements satisfying the given conditions.

(i) $a \in Y$ but $a^2 \notin Y$ (ii) $a + 1 = 6, a \in Y$

(iii) a is less than 6 and $a \in Y$

10. A, B and C are subsets of Universal Set U. If $A = \{2, 4, 6, 8, 12, 20\}$

$B = \{3, 6, 9, 12, 15\}$, $C = \{5, 10, 15, 20\}$ and U is the set of all whole numbers, draw a Venn diagram showing the relation of U, A, B and C.

11. Let U be the set of all boys and girls in a school, G be the set of all girls in the school, B be the set of all boys in the school, and S be the set of all students in the school who take swimming. Some, but not all, students in the school take swimming. Draw a Venn diagram showing one of the possible interrelationship among sets U, G, B and S.

12. For all sets A, B and C, show that $(A - B) \cap (C - B) = A - (B \cup C)$

Determine whether each of the statement in Exercises 13 – 17 is true or false. Justify your answer.

13. For all sets A and B, $(A - B) \cup (A \cap B) = A$

14. For all sets A, B and C, $A - (B - C) = (A - B) - C$

15. For all sets A, B and C, if $A \subset B$, then $A \cap C \subset B \cap C$

16. For all sets A, B and C, if $A \subset B$, then $A \cup C \subset B \cup C$

17. For all sets A, B and C, if $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.

Using properties of sets prove the statements given in Exercises 18 to 22

18. For all sets A and B, $A \cup (B - A) = A \cup B$

19. For all sets A and B, $A - (A - B) = A \cap B$

20. For all sets A and B, $A - (A \cap B) = A - B$

21. For all sets A and B, $(A \cup B) - B = A - B$

22. Let T = $\left\{x \,\middle|\, \frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}\right\}$. Is T an empty set? Justify your answer.

Long Answer Type

23. Let A, B and C be sets. Then show that

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

24. Out of 100 students; 15 passed in English, 12 passed in Mathematics, 8 in Science, 6 in English and Mathematics, 7 in Mathematics and Science; 4 in English and Science; 4 in all the three. Find how many passed

(i) in English and Mathematics but not in Science

(ii) in Mathematics and Science but not in English

(iii) in Mathematics only

(iv) in more than one subject only

25. In a class of 60 students, 25 students play cricket and 20 students play tennis, and 10 students play both the games. Find the number of students who play neither?

26. In a survey of 200 students of a school, it was found that 120 study Mathematics, 90 study Physics and 70 study Chemistry, 40 study Mathematics and Physics, 30 study Physics and Chemistry, 50 study Chemistry and Mathematics and 20 none of these subjects. Find the number of students who study all the three subjects.

27. In a town of 10,000 families it was found that 40% families buy newspaper A, 20% families buy newspaper B, 10% families buy newspaper C, 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three newspapers. Find

(a) The number of families which buy newspaper A only.

(b) The number of families which buy none of A, B and C

28. In a group of 50 students, the number of students studying French, English, Sanskrit were found to be as follows:
French = 17, English = 13, Sanskrit = 15
French and English = 09, English and Sanskrit = 4

French and Sanskrit = 5, English, French and Sanskrit = 3. Find the number of students who study

(i) French only

(ii) English only

(iii) Sanskrit only

(iv) English and Sanskrit but not French

(v) French and Sanskrit but not English

(vi) French and English but not Sanskrit

(vii) at least one of the three languages

(viii) none of the three languages

Objective Type Questions

Choose the correct answers from the given four options in each Exercises 29 to 43 (M.C.Q.).

29. Suppose $A_1, A_2, ..., A_{30}$ are thirty sets each having 5 elements and $B_1, B_2, ..., B_n$ are n sets each with 3 elements, let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each element of S belongs to exactly 10 of the A_i's and exactly 9 of the B_j'S. then n is equal to

(A) 15 (B) 3 (C) 45 (D) 35

30. Two finite sets have m and n elements. The number of subsets of the first set is 112 more than that of the second set. The values of m and n are, respectively,

(A) 4, 7 (B) 7, 4 (C) 4, 4 (D) 7, 7

31. The set $(A \cap B')' \cup (B \cap C)$ is equal to

(A) $A' \cup B \cup C$ (B) $A' \cup B$ (C) $A' \cup C'$ (D) $A' \cap B$

32. Let F_1 be the set of parallelograms, F_2 the set of rectangles, F_3 the set of rhombuses, F_4 the set of squares and F_5 the set of trapeziums in a plane. Then F_1 may be equal to

(A) $F_2 \cap F_3$ (B) $F_3 \cap F_4$

(C) $F_2 \cup F_5$ (D) $F_2 \cup F_3 \cup F_4 \cup F_1$

33. Let S = set of points inside the square, T = the set of points inside the triangle and C = the set of points inside the circle. If the triangle and circle intersect each other and are contained in a square. Then

(A) $S \cap T \cap C = \phi$ (B) $S \cup T \cup C = C$

(C) $S \cup T \cup C = S$ (D) $S \cup T = S \cap C$

34. Let R be set of points inside a rectangle of sides a and b ($a, b > 1$) with two sides along the positive direction of x-axis and y-axis. Then

(A) $R = \{(x, y) : 0 \le x \le a, 0 \le y \le b\}$

(B) $R = \{(x, y) : 0 \le x < a, 0 \le y \le b\}$

(C) $R = \{(x, y) : 0 \le x \le a, 0 < y < b\}$

(D) $R = \{(x, y) : 0 < x < a, 0 < y < b\}$

35. In a class of 60 students, 25 students play cricket and 20 students play tennis, and 10 students play both the games. Then, the number of students who play neither is

(A) 0 (B) 25 (C) 35 (D) 45

36. In a town of 840 persons, 450 persons read Hindi, 300 read English and 200 read both. Then the number of persons who read neither is

(A) 210 (B) 290 (C) 180 (D) 260

37. If $X = \{8^n - 7n - 1 \mid n \in \mathbf{N}\}$ and $Y = \{49n - 49 \mid n \in \mathbf{N}\}$. Then

(A) $X \subset Y$ (B) $Y \subset X$ (C) $X = Y$ (D) $X \cap Y = \phi$

38. A survey shows that 63% of the people watch a News Channel whereas 76% watch another channel. If x% of the people watch both channel, then

(A) $x = 35$ (B) $x = 63$ (C) $39 \le x \le 63$ (D) $x = 39$

39. If sets A and B are defined as

$A = \{(x, y) \mid y = \frac{1}{x}, 0 \ne x \in \mathbf{R}\}$ $\quad B = \{(x, y) \mid y = -x, x \in \mathbf{R}\}$, then

(A) $A \cap B = A$ (B) $A \cap B = B$ (C) $A \cap B = \phi$ (D) $A \cup B = A$

40. If A and B are two sets, then $A \cap (A \cup B)$ equals

(A) A (B) B (C) ϕ (D) $A \cap B$

41. If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$ $B = \{2, 4, ..., 18\}$ and **N** the set of natural numbers is the universal set, then $A' \cup ((A \cup B) \cap B')$ is

(A) ϕ (B) **N** (C) A (D) B

42. Let $S = \{x \mid x$ is a positive multiple of 3 less than 100$\}$

$P = \{x \mid x$ is a prime number less than 20$\}$. Then $n(S) + n(P)$ is

(A) 34 (B) 31 (C) 33 (D) 30

43. If X and Y are two sets and X' denotes the complement of X, then $X \cap (X \cup Y)'$ is equal to

(A) X (B) Y (C) ϕ (D) $X \cap Y$

Fill in the blanks in each of the Exercises from 44 to 51 :

44. The set $\{x \in \mathbf{R} : 1 \le x < 2\}$ can be written as _______________.

45. When $A = \phi$, then number of elements in P(A) is _______________.

46. If A and B are finite sets such that $A \subset B$, then $n(A \cup B) =$ _______________.

47. If A and B are any two sets, then $A - B$ is equal to _______________.

48. Power set of the set $A = \{1, 2\}$ is _______________.

49. Given the sets $A = \{1, 3, 5\}$. $B = \{2, 4, 6\}$ and $C = \{0, 2, 4, 6, 8\}$. Then the universal set of all the three sets A, B and C can be _______________.

50. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 5\}$, $B = \{2, 4, 6, 7\}$ and $C = \{2, 3, 4, 8\}$. Then

(i) $(B \cup C)'$ is _______________. (ii) $(C - A)'$ is _______________.

51. For all sets A and B, $A - (A \cap B)$ is equal to _______________.

52. Match the following sets for all sets A, B and C

(i)	$((A' \cup B') - A)'$	(a)	$A - B$
(ii)	$[B' \cup (B' - A)]'$	(b)	A
(iii)	$(A - B) - (B - C)$	(c)	B
(iv)	$(A - B) \cap (C - B)$	(d)	$(A \times B) \cap (A \times C)$
(v)	$A \times (B \cap C)$	(e)	$(A \times B) \cup (A \times C)$
(vi)	$A \times (B \cup C)$	(f)	$(A \cap C) - B$

State True or False for the following statements in each of the Exercises from 53 to 58 :

53. If A is any set, then $A \subset A$

54. Given that $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and if $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then $B \not\subset M$

55. The sets $\{1, 2, 3, 4\}$ and $\{3, 4, 5, 6\}$ are equal.

56. $\mathbf{Q} \cup \mathbf{Z} = \mathbf{Q}$, where **Q** is the set of rational numbers and **Z** is the set of integers.

57. Let sets R and T be defined as

$R = \{x \in \mathbf{Z} \mid x \text{ is divisible by } 2\}$

$T = \{x \in \mathbf{Z} \mid x \text{ is divisible by } 6\}$. Then $T \subset R$

58. Given $A = \{0, 1, 2\}$, $B = \{x \in \mathbf{R} \mid 0 \le x \le 2\}$. Then $A = B$.

Chapter 2

RELATIONS AND FUNCTIONS

2.1 Overview

This chapter deals with linking pair of elements from two sets and then introduce relations between the two elements in the pair. Practically in every day of our lives, we pair the members of two sets of numbers. For example, each hour of the day is paired with the local temperature reading by T.V. Station's weatherman, a teacher often pairs each set of score with the number of students receiving that score to see more clearly how well the class has understood the lesson. Finally, we shall learn about special relations called functions.

2.1.1 *Cartesian products of sets*

Definition : Given two non-empty sets A and B, the set of all ordered pairs (x, y), where $x \in$ A and $y \in$ B is called Cartesian product of A and B; symbolically, we write

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

If $\quad A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then

$$A \times B = \{(1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5)\}$$

and $\quad B \times A = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$

(i) Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal, i.e. $(x, y) = (u, v)$ if and only if $x = u, y = v$.

(ii) If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = p \times q$.

(iii) $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here (a, b, c) is called an ordered triplet.

2.1.2 *Relations* A Relation R from a non-empty set A to a non empty set B is a subset of the Cartesian product set A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B.

The set of all first elements in a relation R, is called the domain of the relation R, and the set of all second elements called images, is called the range of R.

For example, the set $R = \{(1, 2), (-2, 3), (\frac{1}{2}, 3)\}$ is a relation; the domain of $R = \{1, -2, \frac{1}{2}\}$ and the range of $R = \{2, 3\}$.

(i) A relation may be represented either by the Roster form or by the set builder form, or by an arrow diagram which is a visual representation of a relation.

(ii) If n (A) = p, n (B) = q; then the n (A × B) = pq and the total number of possible relations from the set A to set B = 2^{pq}.

2.1.3 *Functions* A relation f from a set A to a set B is said to be **function** if every element of set A has one and only one image in set B.

In other words, a function f is a relation such that no two pairs in the relation has the same first element.

The notation $f : X \to Y$ means that f is a function from X to Y. X is called the **domain** of f and Y is called the **co-domain** of f. Given an element $x \in X$, there is a unique element y in Y that is related to x. The unique element y to which f relates x is denoted by $f(x)$ and is called f of x, or the **value of f at x**, or the *image of x under f*.

The set of all values of $f(x)$ taken together is called the **range of f** or image of X under f. Symbolically.

$$\text{range of } f = \{ y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}$$

Definition : A function which has either **R** or one of its subsets as its range, is called a real valued function. Further, if its domain is also either **R** or a subset of **R**, it is called a real function.

2.1.4 *Some specific types of functions*

(i) **Identity function:**

The function $f : \mathbf{R} \to \mathbf{R}$ defined by $y = f(x) = x$ for each $x \in \mathbf{R}$ is called the **identity function.** Domain of $f = \mathbf{R}$

Range of $f = \mathbf{R}$

(ii) **Constant function:** The function $f : \mathbf{R} \to \mathbf{R}$ defined by $y = f(x) = C$, $x \in \mathbf{R}$, where C is a constant $\in \mathbf{R}$, is a **constant function**.

Domain of $f = \mathbf{R}$

Range of $f = \{C\}$

(iii) **Polynomial function:** A real valued function $f : \mathbf{R} \to \mathbf{R}$ defined by $y = f(x) = a_0 + a_1 x + ...+ a_n x^n$, where $n \in \mathbf{N}$, and $a_0, a_1, a_2 ... a_n \in \mathbf{R}$, for each $x \in \mathbf{R}$, is called Polynomial functions.

(iv) **Rational function:** These are the real functions of the type $\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$. For example $f : \mathbf{R} - \{-2\} \to \mathbf{R}$ defined by $f(x) = \dfrac{x+1}{x+2}$, $\forall x \in \mathbf{R} - \{-2\}$ is a rational function.

(v) **The Modulus function:** The real function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$\forall x \in \mathbf{R}$ is called the modulus function.

Domain of $f = \mathbf{R}$
Range of $f = \mathbf{R}^+ \cup \{0\}$

(vi) **Signum function:** The real function
$f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is called the **signum function**. Domain of $f = \mathbf{R}$, Range of $f = \{1, 0, -1\}$

(vii) **Greatest integer function:** The real function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = [x]$, $x \in \mathbf{R}$ assumes the value of the greatest integer less than or equal to x, is called the **greatest integer function**.

Thus $f(x) = [x] = -1$ for $-1 \leq x < 0$
$f(x) = [x] = 0$ for $0 \leq x < 1$
$[x] = 1$ for $1 \leq x < 2$
$[x] = 2$ for $2 \leq x < 3$ and so on

2.1.5 *Algebra of real functions*

(i) ***Addition of two real functions***

Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be any two real functions, where $X \in \mathbf{R}$.
Then we define $(f + g) : X \to \mathbf{R}$ by $(f + g)(x) = f(x) + g(x)$, for all $x \in X$.

(ii) ***Subtraction of a real function from another***

Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be any two real functions, where $X \subseteq \mathbf{R}$.

Then, we define $(f - g) : X \to \mathbf{R}$ by $(f - g)(x) = f(x) - g(x)$, for all $x \in X$.

(iii) ***Multiplication by a Scalar***

Let $f : X \to \mathbf{R}$ be a real function and α be any scalar belonging to $\mathbf{R}$. Then the product αf is function from X to $\mathbf{R}$ defined by $(\alpha f)(x) = \alpha f(x)$, $x \in X$.

(iv) ***Multiplication of two real functions***

Let $f : X \to \mathbf{R}$ and $g : x \to \mathbf{R}$ be any two real functions, where $X \subseteq \mathbf{R}$. Then product of these two functions i.e. $f\,g : X \to \mathbf{R}$ is defined by $(f\,g)(x) = f(x)\ g(x)\ \forall\ x \in X$.

(v) ***Quotient of two real function***

Let f and g be two real functions defined from $X \to \mathbf{R}$. The quotient of f by g denoted by $\frac{f}{g}$ is a function defined from $X \to \mathbf{R}$ as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ \text{provided } g(x) \neq 0,\ x \in X.$$

Note Domain of sum function $f + g$, difference function $f - g$ and product function fg.

$$= \{x : x \in D_f \cap D_g\}$$

where D_f = Domain of function f

D_g = Domain of function g

Domain of quotient function $\frac{f}{g}$

$$= \{x : x \in D_f \cap D_g \text{ and } g(x) \neq 0\}$$

2.2 Solved Examples

Short Answer Type

Example 1 Let A = {1, 2, 3, 4} and B = {5, 7, 9}. Determine

(i) A × B (ii) B × A

(iii) Is A × B = B × A ? (iv) Is n (A × B) = n (B × A) ?

Solution Since A = {1, 2, 3, 4} and B = {5, 7, 9}. Therefore,

(i) A × B = {(1, 5), (1, 7), (1, 9), (2, 5), (2, 7),
(2, 9), (3, 5), (3, 7), (3, 9), (4, 5), (4, 7), (4, 9)}

(ii) B × A = {(5, 1), (5, 2), (5, 3), (5, 4), (7, 1), (7, 2),
(7, 3), (7, 4), (9, 1), (9, 2), (9, 3), (9, 4)}

(iii) No, A × B ≠ B × A. Since A × B and B × A do not have exactly the same ordered pairs.

(iv) n (A × B) = n (A) × n (B) = 4 × 3 = 12

n (B × A) = n (B) × n (A) = 4 × 3 = 12

Hence $\quad n$ (A × B) = n (B × A)

Example 2 Find x and y if:

(i) $(4x + 3, y) = (3x + 5, -2)$ (ii) $(x - y, x + y) = (6, 10)$

Solution

(i) Since $(4x + 3, y) = (3x + 5, -2)$, so

$4x + 3 = 3x + 5$

or $\quad x = 2$

and $\quad y = -2$

(ii) $x - y = 6$

$x + y = 10$

$\therefore \quad 2x = 16$

or $\quad x = 8$

$8 - y = 6$

$\therefore \quad y = 2$

Example 3 If A = {2, 4, 6, 9} and B = {4, 6, 18, 27, 54}, $a \in$ A, $b \in$ B, find the set of ordered pairs such that 'a' is factor of 'b' and $a < b$.

Solution Since A = {2, 4, 6, 9}

B = {4, 6, 18, 27, 54},

we have to find a set of ordered pairs (a, b) such that a is factor of b and $a < b$.

Since 2 is a factor of 4 and 2 < 4.

So (2, 4) is one such ordered pair.

Similarly, (2, 6), (2, 18), (2, 54) are other such ordered pairs. Thus the required set of ordered pairs is

{(2, 4), (2, 6), (2, 18), (2, 54), (6, 18), (6, 54,), (9, 18), (9, 27), (9, 54)}.

Example 4 Find the domain and range of the relation R given by

$$R = \{(x, y) : y = x + \frac{6}{x}; \text{ where } x, y \in \mathbf{N} \text{ and } x < 6\}.$$

Solution When $x = 1$, $y = 7 \in \mathbf{N}$, so $(1, 7) \in$ R. Again for,

$$x = 2 \,.\; y = 2 + \frac{6}{2} = 2 + 3 = 5 \in \mathbf{N}, \text{ so } (2, 5) \in \text{ R. Again for}$$

$x = 3,\ y = 3 + \frac{6}{3} = 3 + 2 = 5 \in \mathbf{N}$, $(3, 5) \in$ R. Similarly for $x = 4$

$y = 4 + \frac{6}{4} \notin \mathbf{N}$ and for $x = 5,\ y = 5 + \frac{6}{5} \notin \mathbf{N}$

Thus R = {(1, 7), (2, 5), (3, 5)}, where Domain of R = {1, 2, 3}

Range of R = {7, 5}

Example 5 Is the following relation a function? Justify your answer

(i) $R_1 = \{(2, 3), (\frac{1}{2}, 0), (2, 7), (-4, 6)\}$

(ii) $R_2 = \{(x, |x|) \mid x \text{ is a real number}\}$

Solution

Since $(2, 3)$ and $(2, 7) \in R_1$

$\Rightarrow$ $R_1(2) = 3$ and $R_1(2) = 7$

So $R_1(2)$ does not have a unique image. Thus R_1 is not a function.

(iii) $R_2 = \{(x, |x|) / x \in \mathbf{R}\}$

For every $x \in \mathbf{R}$ there will be unique image as $|x| \in \mathbf{R}$.

Therefore R_2 is a function.

Example 6 Find the domain for which the functions

$f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.

Solution

For $f(x) = g(x)$

$\Rightarrow$ $2x^2 - 1 = 1 - 3x$

$\Rightarrow$ $2x^2 + 3x - 2 = 0$

$\Rightarrow$ $2x^2 + 4x - x - 2 = 0$

$\Rightarrow$ $2x(x + 2) - 1(x + 2) = 0$

$\Rightarrow$ $(2x - 1)(x + 2) = 0$

Thus domain for which the function $f(x) = g(x)$ is $\left\{\frac{1}{2}, -2\right\}$.

Example 7 Find the domain of each of the following functions.

(i) $f(x) = \dfrac{x}{x^2 + 3x + 2}$ (ii) $f(x) = [x] + x$

Solution

(i) f is a rational function of the form $\frac{g(x)}{h(x)}$, where $g(x) = x$ and $h(x) = x^2 + 3x + 2$.

Now $h(x) \neq 0 \Rightarrow x^2 + 3x + 2 \neq 0 \Rightarrow (x + 1)(x + 2) \neq 0$ and hence domain of the given function is R – {– 1, – 2}.

(ii) $f(x) = [x] + x$, i.e., $f(x) = h(x) + g(x)$

where $h(x) = [x]$ and $g(x) = x$

The domain of $h = \mathbf{R}$

and the domain of $g = \mathbf{R}$. Therefore

Domain of $f = \mathbf{R}$

Example 8 Find the range of the following functions given by

(i) $\frac{|x-4|}{x-4}$ (ii) $\sqrt{16-x^2}$

Solution

(i) $f(x) = \frac{|x-4|}{x-4} = \begin{cases} \frac{x-4}{x-4} = 1, & x > 4 \\ \frac{-(x-4)}{x-4} = -1, & x < 4 \end{cases}$

Thus the range of $\frac{|x-4|}{x-4} = \{1, -1\}$.

(ii) The domain of f, where $f(x) = \sqrt{16-x^2}$ is given by [– 4, 4].

For the range, let $y = \sqrt{16-x^2}$

then $y^2 = 16 - x^2$

or $x^2 = 16 - y^2$

Since $x \in [-4, 4]$

Thus range of $f = [0, 4]$

Example 9 Redefine the function which is given by

$$f(x) = |x-1| + |1+x|, \quad -2 \leq x \leq 2$$

Solution $f(x) = |x-1| + |1+x|,\ -2 \le x \le 2$

$$= \begin{cases} -x+1\ -1-x, & -2 \le x < -1 \\ -x+1\ +x+1, & -1 \le x < 1 \\ x-1\ +1+x, & 1 \le x \le 2 \end{cases}$$

$$= \begin{cases} -2x\ , & -2 \le x < -1 \\ 2, & -1 \le x < 1 \\ 2x, & 1 \le x \le 2 \end{cases}$$

Example 10 Find the domain of the function f given by $f(x) = \dfrac{1}{\sqrt{[x]^2-[x]-6}}$

Solution Given that $f(x) = \dfrac{1}{\sqrt{[x]^2-[x]-6}}$, f is defined if $[x]^2 - [x] - 6 > 0$.

or $([x]-3)([x]+2) > 0$,

$\Rightarrow$ $[x] < -2$ or $[x] > 3$

$\Rightarrow$ $x < -2$ or $x \ge 4$

Hence Domain $= (-\infty, -2) \cup [4, \infty)$.

Objective Type Questions

Choose the correct answer out of the four given possible answers (M.C.Q.)

Example 11 The domain of the function f defined by $f(x) = \dfrac{1}{\sqrt{x-|x|}}$ is

(A) $\mathbf{R}$ (B) $\mathbf{R}^+$

(C) $\mathbf{R}^-$ (D) None of these

Solution The correct answer is (D). Given that $f(x) = \dfrac{1}{\sqrt{x-|x|}}$

where $x - |x| = \begin{cases} x-x=0 & \text{if } x \ge 0 \\ 2x & \text{if } x < 0 \end{cases}$

Thus $\frac{1}{\sqrt{x-|x|}}$ is not defined for any $x \in \mathbf{R}$.

Hence f is not defined for any $x \in \mathbf{R}$, i.e. Domain of f is none of the given options.

Example 12 If $f(x) = x^3 - \frac{1}{x^3}$, then $f(x) + f(\frac{1}{x})$ is equal to

(A) $2x^3$ (B) $2\frac{1}{x^3}$ (C) 0 (D) 1

Solution The correct choice is C.

Since $$f(x) = x^3 - \frac{1}{x^3}$$

$$f\left(\frac{1}{x}\right) = \frac{1}{x^3} - \frac{1}{\frac{1}{x^3}} = \frac{1}{x^3} - x^3$$

Hence, $$f(x) + f\left(\frac{1}{x}\right) = x^3 - \frac{1}{x^3} + \frac{1}{x^3} - x^3 = 0$$

Example 13 Let A and B be any two sets such that $n(\text{B}) = p$, $n(\text{A}) = q$ then the total number of functions $f : \text{A} \to \text{B}$ is equal to __________.

Solution Any element of set A, say x_i can be connected with the element of set B in p ways. Hence, there are exactly p^q functions.

Example 14 Let f and g be two functions given by

$f = \{(2, 4), (5, 6), (8, -1), (10, -3)\}$

$g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, -5)\}$ then. Domain of $f + g$ is __________

Solution Since Domain of $f = \text{D}_f = \{2, 5, 8, 10\}$ and Domain of $g = \text{D}_g = \{2, 7, 8, 10, 11\}$, therefore the domain of $f + g = \{x \mid x \in \text{D}_f \cap \text{D}_g\} = \{2, 8, 10\}$

2.3 EXERCISE

Short Answer Type

1. Let A = {–1, 2, 3} and B = {1, 3}. Determine
   - (i) A × B
   - (ii) B × A
   - (iii) B × B
   - (iv) A × A

2. If P = $\{x : x < 3, x \in \mathbf{N}\}$, Q = $\{x : x \le 2, x \in \mathbf{W}\}$. Find (P ∪ Q) × (P ∩ Q), where **W** is the set of whole numbers.

3. If A = $\{x : x \in \mathbf{W}, x < 2\}$ B = $\{x : x \in \mathbf{N}, 1 < x < 5\}$ C = {3, 5} find

(i) A × (B ∩ C) (ii) A × (B ∪ C)

4. In each of the following cases, find a and b.

(i) $(2a + b, a - b) = (8, 3)$ (ii) $\left(\frac{a}{4}, a-2b\right) = (0, 6 + b)$

5. Given A = {1, 2, 3, 4, 5}, S = $\{(x, y) : x \in A, y \in A\}$. Find the ordered pairs which satisfy the conditions given below:

(i) $x + y = 5$ (ii) $x + y < 5$ (iii) $x + y > 8$

6. Given R = $\{(x, y) : x, y \in \mathbf{W}, x^2 + y^2 = 25\}$. Find the domain and Range of R.

7. If $R_1 = \{(x, y) \mid y = 2x + 7$, where $x \in \mathbf{R}$ and $-5 \le x \le 5\}$ is a relation. Then find the domain and Range of R_1.

8. If $R_2 = \{(x, y) \mid x$ and y are integers and $x^2 + y^2 = 64\}$ is a relation. Then find R_2.

9. If $R_3 = \{(x, |x|) \mid x$ is a real number$\}$ is a relation. Then find domain and range of R_3.

10. Is the given relation a function? Give reasons for your answer.

(i) $h = \{(4, 6), (3, 9), (-11, 6), (3, 11)\}$

(ii) $f = \{(x, x) \mid x$ is a real number$\}$

(iii) $g = \left\{\left(n, \frac{1}{n}\right) \mid n \text{ is a positive integer}\right\}$

(iv) $s = \{(n, n^2) \mid n$ is a positive integer$\}$

(v) $t = \{(x, 3) \mid x$ is a real number.

11. If f and g are real functions defined by $f(x) = x^2 + 7$ and $g(x) = 3x + 5$, find each of the following

(a) $f(3) + g(-5)$ (b) $f\left(\frac{1}{2}\right) \times g(14)$

(c) $f(-2) + g(-1)$ (d) $f(t) - f(-2)$

(e) $\frac{f(t) - f(5)}{t - 5}$, if $t \ne 5$

12. Let f and g be real functions defined by $f(x) = 2x + 1$ and $g(x) = 4x - 7$.

(a) For what real numbers x, $f(x) = g(x)$?

(b) For what real numbers x, $f(x) < g(x)$?

13. If f and g are two real valued functions defined as $f(x) = 2x + 1$, $g(x) = x^2 + 1$, then find.

(i) $f + g$ (ii) $f - g$ (iii) fg (iv) $\dfrac{f}{g}$

14. Express the following functions as set of ordered pairs and determine their range.
$f : \text{X} \to \mathbf{R}$, $f(x) = x^3 + 1$, where X = {–1, 0, 3, 9, 7}

15. Find the values of x for which the functions
$f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal

Long Answer Type

16. Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? Justify. If this is described by the relation, $g(x) = \alpha x + \beta$, then what values should be assigned to α and β?

17. Find the domain of each of the following functions given by

(i) $f(x) = \dfrac{1}{\sqrt{1-\cos x}}$ (ii) $f(x) = \dfrac{1}{\sqrt{x+|x|}}$

(iii) $f(x) = x\,|x|$ (iv) $f(x) = \dfrac{x^3 - x + 3}{x^2 - 1}$

(v) $f(x) = \dfrac{3x}{2x-8}$

18. Find the range of the following functions given by

(i) $f(x) = \dfrac{3}{2-x^2}$ (ii) $f(x) = 1 - |x-2|$

(iii) $f(x) = |x-3|$ (iv) $f(x) = 1 + 3\cos 2x$

(**Hint :** $-1 \le \cos 2x \le 1 \Rightarrow -3 \le 3\cos 2x \le 3 \Rightarrow -2 \le 1 + 3\cos 2x \le 4$)

19. Redefine the function $f(x) = |x-2| + |2+x|$, $-3 \le x \le 3$

20. If $f(x) = \dfrac{x-1}{x+1}$, then show that

(i) $f\left(\dfrac{1}{x}\right) = -f(x)$ (ii) $f\left(-\dfrac{1}{x}\right) = \dfrac{-1}{f(x)}$

21. Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined in the domain $R^+ \cup \{0\}$. Find

(i) $(f + g)(x)$ (ii) $(f - g)(x)$

(iii) $(fg)(x)$ (iv) $\left(\dfrac{f}{g}\right)(x)$

22. Find the domain and Range of the function $f(x) = \dfrac{1}{\sqrt{x-5}}$.

23. If $f(x) = y = \dfrac{ax-b}{cx-a}$, then prove that $f(y) = x$.

Objective Type Questions

Choose the correct answers in Exercises from 24 to 35 (M.C.Q.)

24. Let n (A) $= m$, and n (B) $= n$. Then the total number of non-empty relations that can be defined from A to B is

(A) m^n (B) $n^m - 1$

(C) $mn - 1$ (D) $2^{mn} - 1$

25. If $[x]^2 - 5[x] + 6 = 0$, where [.] denote the greatest integer function, then

(A) $x \in [3, 4]$ (B) $x \in (2, 3]$

(C) $x \in [2, 3]$ (D) $x \in [2, 4)$

26. Range of $f(x) = \dfrac{1}{1-2\cos x}$ is

(A) $\left[\dfrac{1}{3}, 1\right]$ (B) $\left[-1, \dfrac{1}{3}\right]$

(C) $(-\infty, -1] \cup \left[\dfrac{1}{3}, \infty\right)$ (D) $\left[-\dfrac{1}{3}, 1\right]$

27. Let $f(x) = \sqrt{1+x^2}$, then

(A) $f(xy) = f(x) \,.\, f(y)$ (B) $f(xy) \geq f(x) \,.\, f(y)$

(C) $f(xy) \leq f(x) \,.\, f(y)$ (D) None of these

[**Hint :** find $f(xy) = \sqrt{1+x^2y^2}$, $f(x) \,.\, f(y) = \sqrt{1+x^2y^2+x^2+y^2}$]

28. Domain of $\sqrt{a^2 - x^2}$ $(a > 0)$ is

(A) $(-a, a)$ (B) $[-a, a]$

(C) $[0, a]$ (D) $(-a, 0]$

29. If $f(x) = ax + b$, where a and b are integers, $f(-1) = -5$ and $f(3) = 3$, then a and b are equal to

(A) $a = -3, b = -1$ (B) $a = 2, b = -3$

(C) $a = 0, b = 2$ (D) $a = 2, b = 3$

30. The domain of the function f defined by $f(x) = \sqrt{4-x} + \dfrac{1}{\sqrt{x^2-1}}$ is equal to

(A) $(-\infty, -1) \cup (1, 4]$ (B) $(-\infty, -1] \cup (1, 4]$

(C) $(-\infty, -1) \cup [1, 4]$ (D) $(-\infty, -1) \cup [1, 4)$

31. The domain and range of the real function f defined by $f(x) = \dfrac{4-x}{x-4}$ is given by

(A) Domain = **R**, Range = $\{-1, 1\}$

(B) Domain = **R** – $\{1\}$, Range = **R**

(C) Domain = **R** – $\{4\}$, Range = $\{-1\}$

(D) Domain = **R** – $\{-4\}$, Range = $\{-1, 1\}$

32. The domain and range of real function f defined by $f(x) = \sqrt{x-1}$ is given by

(A) Domain = $(1, \infty)$, Range = $(0, \infty)$

(B) Domain = $[1, \infty)$, Range = $(0, \infty)$

(C) Domain = $[1, \infty)$, Range = $[0, \infty)$

(D) Domain = $[1, \infty)$, Range = $[0, \infty)$

33. The domain of the function f given by $f(x) = \dfrac{x^2+2x+1}{x^2-x-6}$

(A) $\mathbf{R} - \{3, -2\}$ (B) $\mathbf{R} - \{-3, 2\}$

(C) $\mathbf{R} - [3, -2]$ (D) $\mathbf{R} - (3, -2)$

34. The domain and range of the function f given by $f(x) = 2 - |x-5|$ is

(A) Domain = $\mathbf{R}^+$, Range = $(-\infty, 1]$

(B) Domain = $\mathbf{R}$, Range = $(-\infty, 2]$

(C) Domain = $\mathbf{R}$, Range = $(-\infty, 2)$

(D) Domain = $\mathbf{R}^+$, Range = $(-\infty, 2]$

35. The domain for which the functions defined by $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal is

(A) $\left\{-1, \dfrac{4}{3}\right\}$ (B) $\left[-1, \dfrac{4}{3}\right]$

(C) $\left(-1, \dfrac{4}{3}\right)$ (D) $\left[-1, \dfrac{4}{3}\right)$

Fill in the blanks :

36. Let f and g be two real functions given by

$f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 1)\}$

$g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}$

then the domain of $f \cdot g$ is given by __________.

37. Let $f = \{(2, 4), (5, 6), (8, -1), (10, -3)\}$

$g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, 5)\}$

be two real functions. Then Match the following :

(a) $f - g$ (i) $\left\{\left(2, \dfrac{4}{5}\right), \left(8, \dfrac{-1}{4}\right), \left(10, \dfrac{-3}{13}\right)\right\}$

(b) $f + g$ (ii) $\{(2, 20), (8, -4), (10, -39)\}$

(c) $f \cdot g$ (iii) $\{(2, -1), (8, -5), (10, -16)\}$

(d) $\dfrac{f}{g}$ (iv) $\{(2, 9), (8, 3), (10, 10)\}$

State True or False for the following statements given in Exercises 38 to 42 :

38. The ordered pair (5, 2) belongs to the relation $R = \{(x, y) : y = x - 5, x, y \in \mathbf{Z}\}$

39. If $P = \{1, 2\}$, then $P \times P \times P = \{(1, 1, 1), (2, 2, 2), (1, 2, 2), (2, 1, 1)\}$

40. If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$, then $(A \times B) \cup (A \times C)$
$= \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$.

41. If $(x - 2, y + 5) = \left(-2, \dfrac{1}{3}\right)$ are two equal ordered pairs, then $x = 4$, $y = \dfrac{-14}{3}$

42. If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$, then $A = \{a, b\}$, $B = \{x, y\}$

Chapter 3

TRIGONOMETRIC FUNCTIONS

3.1 Overview

3.1.1 The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' which means measuring the sides of a triangle. An angle is the amount of rotation of a revolving line with respect to a fixed line. If the rotation is in clockwise direction the angle is negative and it is positive if the rotation is in the anti-clockwise direction. Usually we follow two types of conventions for measuring angles, i.e., (i) Sexagesimal system (ii) Circular system.

In sexagesimal system, the unit of measurement is degree. If the rotation from the initial to terminal side is $\frac{1}{360}$th of a revolution, the angle is said to have a measure of 1°. The classifications in this system are as follows:

$$1° = 60'$$

$$1' = 60''$$

In circular system of measurement, the unit of measurement is radian. One radian is the angle subtended, at the centre of a circle, by an arc equal in length to the radius of the circle. The length s of an arc PQ of a circle of radius r is given by $s = r\theta$, where θ is the angle subtended by the arc PQ at the centre of the circle measured in terms of radians.

3.1.2 *Relation between degree and radian*

The circumference of a circle always bears a constant ratio to its diameter. This constant ratio is a number denoted by π which is taken approximately as $\frac{22}{7}$ for all practical purpose. The relationship between degree and radian measurements is as follows:

$$2 \text{ right angle} = 180° = \pi \text{ radians}$$

$$1 \text{ radian} = \frac{180°}{\pi} = 57°16' \text{ (approx)}$$

$$1° = \frac{\pi}{180} \text{ radian} = 0.01746 \text{ radians (approx)}$$

3.1.3 *Trigonometric functions*

Trigonometric ratios are defined for acute angles as the ratio of the sides of a right angled triangle. The extension of trigonometric ratios to any angle in terms of radian measure (real numbers) are called trigonometric functions. The signs of trigonometric functions in different quadrants have been given in the following table:

	I	II	III	IV
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
$\operatorname{cosec} x$	+	+	–	–
$\sec x$	+	–	–	+
$\cot x$	+	–	+	–

3.1.4 *Domain and range of trigonometric functions*

Functions	**Domain**	**Range**
sine	$\mathbf{R}$	$[-1, 1]$
cosine	$\mathbf{R}$	$[-1, 1]$
tan	$\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$	$\mathbf{R}$
cot	$\mathbf{R} - \{n\pi : n \in \mathbf{Z}\}$	$\mathbf{R}$
sec	$\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$	$\mathbf{R} - (-1, 1)$
cosec	$\mathbf{R} - \{n\pi : n \in \mathbf{Z}\}$	$\mathbf{R} - (-1, 1)$

3.1.5 *Sine, cosine and tangent of some angles less than 90°*

	0°	**15°**	**18°**	**30°**	**36°**	**45°**	**60°**	**90°**
sine	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

cosine	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\frac{1}{\sqrt{3}}$	$\sqrt{5-2\sqrt{5}}$	1	$\sqrt{3}$	not defined

3.1.6 *Allied or related angles* The angles $\frac{n\pi}{2} \pm \theta$ are called allied or related angles and $\theta \pm n \times 360°$ are called coterminal angles. For general reduction, we have the following rules. The value of any trigonometric function for $(\frac{n\pi}{2} \pm \theta)$ is numerically equal to

(a) the value of the same function if *n* is an even integer with algebraic sign of the function as per the quadrant in which angles lie.

(b) corresponding cofunction of θ if *n* is an odd integer with algebraic sign of the function for the quadrant in which it lies. Here sine and cosine; tan and cot; sec and cosec are cofunctions of each other.

3.1.7 *Functions of negative angles* Let θ be any angle. Then

$$\sin(-\theta) = -\sin\theta,\quad \cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta,\quad \cot(-\theta) = -\cot\theta$$
$$\sec(-\theta) = \sec\theta,\quad \text{cosec}(-\theta) = -\text{cosec}\,\theta$$

3.1.8 *Some formulae regarding compound angles*

An angle made up of the sum or differences of two or more angles is called a compound angle. The basic results in this direction are called trigonometric identies as given below:

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$

(vii) $\cot (A + B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

(viii) $\cot (A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(ix) $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$

(x) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1 = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(xi) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

(xii) $\sin 3A = 3\sin A - 4\sin^3 A$

(xiii) $\cos 3A = 4\cos^3 A - 3\cos A$

(xiv) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$

(xv) $\cos A + \cos B = 2 \cos \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

(xvi) $\cos A - \cos B = 2\sin \dfrac{A + B}{2} \sin \dfrac{B - A}{2}$

(xvii) $\sin A + \sin B = 2\sin \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

(xviii) $\sin A - \sin B = 2\cos \dfrac{A + B}{2} \sin \dfrac{A - B}{2}$

(xix) $2\sin A \cos B = \sin (A + B) + \sin (A - B)$

(xx) $2\cos A \sin B = \sin (A + B) - \sin (A - B)$

(xxi) $2\cos A \cos B = \cos (A + B) + \cos (A - B)$

(xxii) $2\sin A \sin B = \cos (A - B) - \cos (A + B)$

(xxiii) $\sin \dfrac{A}{2} = \pm \sqrt{\dfrac{1 - \cos A}{2}} \quad \left[\begin{array}{l} + \text{ if } \dfrac{A}{2} \text{ lies in quadrants I or II} \\ - \text{ if } \dfrac{A}{2} \text{ lies in III or IV quadrants} \end{array}\right.$

(xxiv) $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$ $\begin{bmatrix} + \text{ if } \frac{A}{2} \text{ lies in I or IV quadrants} \\ - \text{ if } \frac{A}{2} \text{ lies in II or III quadrants} \end{bmatrix}$

(xxv) $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$ $\begin{bmatrix} + \text{ if } \frac{A}{2} \text{ lies in I or III quadrants} \\ - \text{ if } \frac{A}{2} \text{ lies in II or IV quadrants} \end{bmatrix}$

Trigonometric functions of an angle of 18°

Let $\theta = 18°$. Then $2\theta = 90° - 3\theta$

Therefore, $\sin 2\theta = \sin (90° - 3\theta) = \cos 3\theta$

or $\sin 2\theta = 4\cos^3\theta - 3\cos\theta$

Since, $\cos\theta \neq 0$, we get

$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta$ or $4\sin^2\theta + 2\sin\theta - 1 = 0.$

Hence, $\sin\theta = \frac{-2\pm\sqrt{4+16}}{8} = \frac{-1\pm\sqrt{5}}{4}$

Since, $\theta = 18°$, $\sin\theta > 0$, therefore, $\sin 18° = \frac{\sqrt{5}-1}{4}$

Also, $\cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{4}}$

Now, we can easily find cos 36° and sin 36° as follows:

$$\cos 36° = 1 - 2\sin^2 18° = 1 - \frac{6-2\sqrt{5}}{8} = \frac{2+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4}$$

Hence, $\cos 36° = \frac{\sqrt{5}+1}{4}$

Also, $\sin 36° = \sqrt{1-\cos^2 36°} = \sqrt{1-\frac{6+2\sqrt{5}}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$

3.1.9 *Trigonometric equations*

Equations involving trigonometric functions of a variables are called **trigonometric equations**. Equations are called identities, if they are satisfied by all values of the unknown angles for which the functions are defined. The solutions of a trigonometric equations for which $0 \le \theta < 2\pi$ are called **principal solutions**. The expression involving integer n which gives all solutions of a trigonometric equation is called the **general solution**.

General Solution of Trigonometric Equations

(i) If $\sin\theta = \sin\alpha$ for some angle α, then

$\theta = n\pi + (-1)^n\alpha$ for $n \in \mathbf{Z}$, gives general solution of the given equation

(ii) If $\cos\theta = \cos\alpha$ for some angle α, then

$\theta = 2n\pi \pm \alpha$, $n \in \mathbf{Z}$, gives general solution of the given equation

(iii) If $\tan\theta = \tan\alpha$ or $\cot\theta = \cot\alpha$, then

$\theta = n\pi + \alpha$, $n \in \mathbf{Z}$, gives general solution for both equations

(iv) The general value of θ satisfying any of the equations $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$ and

$\tan^2\theta = \tan^2\alpha$ is given by $\theta = n\pi \pm \alpha$

(v) The general value of θ satisfying equations $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha$ simultaneously is given by $\theta = 2n\pi + \alpha$, $n \in \mathbf{Z}$.

(vi) To find the solution of an equation of the form $a\cos\theta + b\sin\theta = c$, we put

$a = r\cos\alpha$ and $b = r\sin\alpha$, so that $r^2 = a^2 + b^2$ and $\tan\alpha = \dfrac{b}{a}$.

Thus we find

$a\cos\theta + b\sin\theta = c$ changed into the form $r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = c$

or $r\cos(\theta - \alpha) = c$ and hence $\cos(\theta - \alpha) = \dfrac{c}{r}$. This gives the solution of the given equation.

Maximum and Minimum values of the expression $A\cos\theta + B\sin\theta$ are $\sqrt{A^2 + B^2}$ and $-\sqrt{A^2 + B^2}$ respectively, where A and B are constants.

3.2 Solved Examples

Short Answer Type

Example 1 A circular wire of radius 3 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 48 cm. Find the angle in degrees which is subtended at the centre of hoop.

Solution Given that circular wire is of radius 3 cm, so when it is cut then its length $= 2\pi \times 3 = 6\pi$ cm. Again, it is being placed along a circular hoop of radius 48 cm. Here, $s = 6\pi$ cm is the length of arc and $r = 48$ cm is the radius of the circle. Therefore, the angle θ, in radian, subtended by the arc at the centre of the circle is given by

$$\theta = \frac{\text{Arc}}{\text{Radius}} = \frac{6\pi}{48} = \frac{\pi}{8} = 22.5^\circ.$$

Example 2 If $A = \cos^2\theta + \sin^4\theta$ for all values of θ, then prove that $\frac{3}{4} \le A \le 1$.

Solution We have $A = \cos^2\theta + \sin^4\theta = \cos^2\theta + \sin^2\theta \sin^2\theta \le \cos^2\theta + \sin^2\theta$

Therefore, $A \le 1$

Also, $A = \cos^2\theta + \sin^4\theta = (1 - \sin^2\theta) + \sin^4\theta$

$$= \left(\sin^2\theta - \frac{1}{2}\right)^2 + \left(1 - \frac{1}{4}\right) = \left(\sin^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \ge \frac{3}{4}$$

Hence, $\frac{3}{4} \le A \le 1$.

Example 3 Find the value of $\sqrt{3}$ cosec 20° – sec 20°

Solution We have

$$\sqrt{3}\ \text{cosec}\ 20^\circ - \sec 20^\circ = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ} = 4\left(\frac{\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ}{2\sin 20^\circ \cos 20^\circ}\right)$$

$$= 4\left(\frac{\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ}{\sin 40^\circ}\right)$$

(Why?)

$$= 4\left(\frac{\sin(60^\circ - 20^\circ)}{\sin 40^\circ}\right) = 4 \qquad \text{(Why?)}$$

Example 4 If θ lies in the second quadrant, then show that

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}+\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=-2\sec\theta$$

Solution We have

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}+\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\frac{1-\sin\theta}{\sqrt{1-\sin^2\theta}}+\frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}}=\frac{2}{\sqrt{\cos^2\theta}}$$

$$=\frac{2}{|\cos\theta|} \qquad \left(\text{Since } \sqrt{\alpha^2}=|\alpha| \text{ for every real number } \alpha\right)$$

Given that θ lies in the second quadrant so $|\cos\theta| = -\cos\theta$ (since $\cos\theta < 0$).

Hence, the required value of the expression is $\dfrac{2}{-\cos\theta}=-2\sec\theta$

Example 5 Find the value of tan 9° – tan 27° – tan 63° + tan 81°

Solution We have tan 9° – tan 27° – tan 63° + tan 81°

= tan 9° + tan 81° – tan 27° – tan 63°

= tan 9° + tan (90° – 9°) – tan 27° – tan (90° – 27°)

= tan 9° + cot 9° – (tan 27° + cot 27°) (1)

Also $\tan 9^\circ + \cot 9^\circ = \dfrac{1}{\sin 9^\circ \cos 9^\circ}=\dfrac{2}{\sin 18^\circ}$ (Why?) (2)

Similarly, $\tan 27^\circ + \cot 27^\circ = \dfrac{1}{\sin 27^\circ \cos 27^\circ}=\dfrac{2}{\sin 54^\circ}=\dfrac{2}{\cos 36^\circ}$ (Why?) (3)

Using (2) and (3) in (1), we get

$$\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ = \frac{2}{\sin 18^\circ}-\frac{2}{\cos 36^\circ}=\frac{2\times 4}{\sqrt{5}-1}-\frac{2\times 4}{\sqrt{5}+1}=4$$

Example 6 Prove that $\dfrac{\sec 8\theta-1}{\sec 4\theta-1}=\dfrac{\tan 8\theta}{\tan 2\theta}$

Solution We have

$$\frac{\sec 8\theta-1}{\sec 4\theta-1}=\frac{(1-\cos 8\theta)\cos 4\theta}{\cos 8\theta\,(1-\cos 4\theta)}$$

$$=\frac{2\sin^2 4\theta\cos 4\theta}{\cos 8\theta\; 2\sin^2 2\theta} \qquad \text{(Why?)}$$

$$= \frac{\sin 4\theta (2\sin 4\theta \cos 4\theta)}{2\cos 8\theta \sin^2 2\theta}$$

$$= \frac{\sin 4\theta \sin 8\theta}{2\cos 8\theta \sin^2 2\theta} \qquad \text{(Why?)}$$

$$= \frac{2\sin 2\theta \cos 2\theta \sin 8\theta}{2\cos 8\theta \sin^2 2\theta}$$

$$= \frac{\tan 8\theta}{\tan 2\theta} \qquad \text{(Why?)}$$

Example 7 Solve the equation $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

Solution We have $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

or $(\sin\theta + \sin 5\theta) + \sin 3\theta = 0$

or $2\sin 3\theta \cos 2\theta + \sin 3\theta = 0$ (Why?)

or $\sin 3\theta (2\cos 2\theta + 1) = 0$

or $\sin 3\theta = 0$ or $\cos 2\theta = -\frac{1}{2}$

When $\sin 3\theta = 0$, then $3\theta = n\pi$ or $\theta = \frac{n\pi}{3}$

When $\cos 2\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$, then $2\theta = 2n\pi \pm \frac{2\pi}{3}$ or $\theta = n\pi \pm \frac{\pi}{3}$

which gives $\theta = (3n+1)\frac{\pi}{3}$ or $\theta = (3n-1)\frac{\pi}{3}$

All these values of θ are contained in $\theta = \frac{n\pi}{3}$, $n \in \mathbf{Z}$. Hence, the required solution set is given by $\{\theta : \theta = \frac{n\pi}{3}, n \in \mathbf{Z}\}$

Example 8 Solve $2\tan^2 x + \sec^2 x = 2$ for $0 \le x \le 2\pi$

Solution Here, $2\tan^2 x + \sec^2 x = 2$

which gives $\tan x = \pm\frac{1}{\sqrt{3}}$

If we take $\tan x = \frac{1}{\sqrt{3}}$, then $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$ (Why?)

Again, if we take $\tan x = \frac{-1}{\sqrt{3}}$, then $x = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$ (Why?)

Therefore, the possible solutions of above equations are

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ where $0 \le x \le 2\pi$

Long Answer Type

Example 9 Find the value of $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$

Solution Write $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$

$$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\left(\pi-\frac{3\pi}{8}\right)\right)\left(1+\cos\left(\pi-\frac{\pi}{8}\right)\right)$$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right) \quad \text{(Why?)}$$

$$= \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$= \frac{1}{4}\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos\frac{3\pi}{4}\right) \quad \text{(Why?)}$$

$$= \frac{1}{4}\left(1-\cos\frac{\pi}{4}\right)\left(1+\cos\frac{\pi}{4}\right) \quad \text{(Why?)}$$

$$= \frac{1}{4}\left(1-\cos^2\frac{\pi}{4}\right) = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8}$$

Example 10 If $x \cos\theta = y \cos\left(\theta + \frac{2\pi}{3}\right) = z \cos\left(\theta + \frac{4\pi}{3}\right)$, then find the value of $xy + yz + zx$.

Solution Note that $xy + yz + zx = xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$.

If we put $x \cos\theta = y \cos(\theta + \frac{2\pi}{3}) = z \cos\left(\theta + \frac{4\pi}{3}\right) = k$ (say).

Then $$x = \frac{k}{\cos\theta},\ y = \frac{k}{\cos\left(\theta+\frac{2\pi}{3}\right)} \text{ and } z = \frac{k}{\cos\left(\theta+\frac{4\pi}{3}\right)}$$

so that $$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{k}\left[\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)\right]$$

$$= \frac{1}{k}[\cos\theta+\cos\theta\cos\frac{2\pi}{3}-\sin\theta\sin\frac{2\pi}{3}$$

$$+ \cos\theta\cos\frac{4\pi}{3}-\sin\theta\sin\frac{4\pi}{3}]$$

$$= \frac{1}{k}[\cos\theta+\cos\theta\left(\frac{-1}{2}\right)-\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta] \text{ (Why?)}$$

$$= \frac{1}{k}\times 0 = 0$$

Hence, $xy + yz + zx = 0$

Example 11 If α and β are the solutions of the equation $a \tan\theta + b \sec\theta = c$, then show that $\tan(\alpha + \beta) = \frac{2ac}{a^2 - c^2}$.

Solution Given that $a\tan\theta + b\sec\theta = c$ or $a\sin\theta + b = c\cos\theta$

Using the identities,

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \text{ and } \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

We have, $$\frac{a\left(2\tan\frac{\theta}{2}\right)}{1+\tan^2\frac{\theta}{2}}+b=\frac{c\left(1-\tan^2\frac{\theta}{2}\right)}{1+\tan^2\frac{\theta}{2}}$$

or $$(b+c)\tan^2\frac{\theta}{2}+2a\tan\frac{\theta}{2}+b-c=0$$

Above equation is quadratic in $\tan\frac{\theta}{2}$ and hence $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are the roots of this equation (Why?). Therefore, $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}=\frac{-2a}{b+c}$ and $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{b-c}{b+c}$ (Why?)

Using the identity $$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=\frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$

We have, $$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=\frac{\frac{-2a}{b+c}}{1-\frac{b-c}{b+c}}=\frac{-2a}{2c}=\frac{-a}{c} \quad \text{... (1)}$$

Again, using another identity

$$\tan 2\left(\frac{\alpha+\beta}{2}\right)=\frac{2\tan\frac{\alpha+\beta}{2}}{1-\tan^2\frac{\alpha+\beta}{2}},$$

We have $$\tan(\alpha+\beta)=\frac{2\left(-\frac{a}{c}\right)}{1-\frac{a^2}{c^2}}=\frac{2ac}{a^2-c^2} \quad \text{[From (1)]}$$

Alternatively, given that $a\tan\theta+b\sec\theta=c$

$\Rightarrow \quad (a \tan\theta - c)^2 = b^2(1 + \tan^2\theta)$

$\Rightarrow \quad a^2 \tan^2\theta - 2ac \tan\theta + c^2 = b^2 + b^2 \tan^2\theta$

$\Rightarrow \quad (a^2 - b^2) \tan^2\theta - 2ac \tan\theta + c^2 - b^2 = 0 \qquad \text{... (1)}$

Since α and β are the roots of the equation (1), so

$$\tan\alpha + \tan\beta = \frac{2ac}{a^2 - b^2} \quad \text{and} \quad \tan\alpha \tan\beta = \frac{c^2 - b^2}{a^2 - b^2}$$

Therefore,
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$= \frac{\dfrac{2ac}{a^2 - b^2}}{\dfrac{c^2 - b^2}{a^2 - b^2}} = \frac{2ac}{a^2 - c^2}$$

Example 12 Show that $2 \sin^2 \beta + 4 \cos(\alpha + \beta) \sin\alpha \sin\beta + \cos 2(\alpha + \beta) = \cos 2\alpha$

Solution LHS $= 2 \sin^2 \beta + 4 \cos(\alpha + \beta) \sin\alpha \sin\beta + \cos 2(\alpha + \beta)$

$= 2 \sin^2 \beta + 4 (\cos\alpha \cos\beta - \sin\alpha \sin\beta) \sin\alpha \sin\beta$
$\quad + (\cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta)$

$= 2 \sin^2 \beta + 4 \sin\alpha \cos\alpha \sin\beta \cos\beta - 4 \sin^2\alpha \sin^2\beta$
$\quad + \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta$

$= 2 \sin^2 \beta + \sin 2\alpha \sin 2\beta - 4 \sin^2\alpha \sin^2\beta + \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta$

$= (1 - \cos 2\beta) - (2 \sin^2\alpha)(2 \sin^2\beta) + \cos 2\alpha \cos 2\beta$ (Why?)

$= (1 - \cos 2\beta) - (1 - \cos 2\alpha)(1 - \cos 2\beta) + \cos 2\alpha \cos 2\beta$

$= \cos 2\alpha$ (Why?)

Example 13 If angle θ is divided into two parts such that the tangent of one part is k times the tangent of other, and ϕ is their difference, then show that

$$\sin\theta = \frac{k+1}{k-1} \sin\phi$$

Solution Let $\theta = \alpha + \beta$. Then $\tan\alpha = k \tan\beta$

or $$\frac{\tan\alpha}{\tan\beta} = \frac{k}{1}$$

Applying componendo and dividendo, we have

$$\frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \frac{k+1}{k-1}$$

or $$\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta} = \frac{k+1}{k-1} \qquad \text{(Why?)}$$

i.e., $$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{k+1}{k-1} \qquad \text{(Why?)}$$

Given that $\alpha - \beta = \phi$ and $\alpha + \beta = \theta$. Therefore,

$$\frac{\sin\theta}{\sin\phi} = \frac{k+1}{k-1} \quad \text{or} \quad \sin\theta = \frac{k+1}{k-1}\sin\phi$$

Example 14 Solve $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$

Solution Divide the given equation by 2 to get

$$\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta=\frac{1}{\sqrt{2}} \quad \text{or} \quad \cos\frac{\pi}{6}\cos\theta+\sin\frac{\pi}{6}\sin\theta=\cos\frac{\pi}{4}$$

or $$\cos\left(\frac{\pi}{6}-\theta\right)=\cos\frac{\pi}{4} \text{ or } \cos\left(\theta-\frac{\pi}{6}\right)=\cos\frac{\pi}{4} \qquad \text{(Why?)}$$

Thus, the solution are given by, i.e., $\theta = 2m\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$

Hence, the solution are

$$\theta = 2m\pi + \frac{\pi}{4} + \frac{\pi}{6} \text{ and } 2m\pi - \frac{\pi}{4} + \frac{\pi}{6}, \text{ i.e., } \theta = 2m\pi + \frac{5\pi}{12} \text{ and } \theta = 2m\pi - \frac{\pi}{12}$$

Objective Type Questions

Choose the correct answer from the given four options against each of the Examples 15 to 19

Example 15 If $\tan\theta = \frac{-4}{3}$, then $\sin\theta$ is

(A) $\frac{-4}{5}$ but not $\frac{4}{5}$ (B) $\frac{-4}{5}$ or $\frac{4}{5}$

(C) $\frac{4}{5}$ but not $-\frac{4}{5}$ (D) None of these

Solution Correct choice is B. Since $\tan\theta = -\frac{4}{3}$ is negative, θ lies either in second quadrant or in fourth quadrant. Thus $\sin\theta = \frac{4}{5}$ if θ lies in the second quadrant or $\sin\theta = -\frac{4}{5}$, if θ lies in the fourth quadrant.

Example 16 If $\sin\theta$ and $\cos\theta$ are the roots of the equation $ax^2 - bx + c = 0$, then a, b and c satisfy the relation.

(A) $a^2 + b^2 + 2ac = 0$ (B) $a^2 - b^2 + 2ac = 0$

(C) $a^2 + c^2 + 2ab = 0$ (D) $a^2 - b^2 - 2ac = 0$

Solution The correct choice is (B). Given that $\sin\theta$ and $\cos\theta$ are the roots of the equation $ax^2 - bx + c = 0$, so $\sin\theta + \cos\theta = \frac{b}{a}$ and $\sin\theta\cos\theta = \frac{c}{a}$ (Why?)

Using the identity $(\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$, we have

$$\frac{b^2}{a^2} = 1 + \frac{2c}{a} \text{ or } a^2 - b^2 + 2ac = 0$$

Example 17 The greatest value of $\sin x \cos x$ is

(A) 1 (B) 2 (C) $\sqrt{2}$ (D) $\frac{1}{2}$

Solution (D) is the correct choice, since

$$\sin x \cos x = \frac{1}{2}\sin 2x \le \frac{1}{2}, \text{ since } |\sin 2x| \le 1.$$

Eaxmple 18 The value of $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$ is

(A) $\frac{-3}{16}$ (B) $\frac{5}{16}$ (C) $\frac{3}{16}$ (D) $\frac{1}{16}$

Solution Correct choice is (C). Indeed sin 20° sin 40° sin 60° sin 80°.

$$= \frac{\sqrt{3}}{2} \sin 20° \sin (60° - 20°) \sin (60° + 20°) \text{ (since } \sin 60° = \frac{\sqrt{3}}{2}\text{)}$$

$$= \frac{\sqrt{3}}{2} \sin 20° [\sin^2 60° - \sin^2 20°] \qquad \text{(Why?)}$$

$$= \frac{\sqrt{3}}{2} \sin 20° [\frac{3}{4} - \sin^2 20°]$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{4} [3\sin 20° - 4\sin^3 20°]$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{4} (\sin 60°) \qquad \text{(Why?)}$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{4} \times \frac{\sqrt{3}}{2} = \frac{3}{16}$$

Example 19 The value of $\cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5}$ is

(A) $\frac{1}{16}$ (B) 0 (C) $\frac{-1}{8}$ (D) $\frac{-1}{16}$

Solution (D) is the correct answer. We have

$$\cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5}$$

$$= \frac{1}{2\sin\frac{\pi}{5}} 2\sin\frac{\pi}{5} \cos\frac{\pi}{5} \cos\frac{2\pi}{5} \cos\frac{4\pi}{5} \cos\frac{8\pi}{5}$$

$$= \frac{1}{2\sin\frac{\pi}{5}} \sin\frac{2\pi}{5} \cos\frac{2\pi}{5} \cos\frac{4\pi}{5} \cos\frac{8\pi}{5} \qquad \text{(Why?)}$$

$$= \frac{1}{4\sin\frac{\pi}{5}} \sin\frac{4\pi}{5} \cos\frac{4\pi}{5} \cos\frac{8\pi}{5} \qquad \text{(Why?)}$$

$$=\frac{1}{8\sin\frac{\pi}{5}}\sin\frac{8\pi}{5}\cos\frac{8\pi}{5} \qquad \text{(Why?)}$$

$$=\frac{\sin\frac{16\pi}{5}}{16\sin\frac{\pi}{5}}=\frac{\sin\left(3\pi+\frac{\pi}{5}\right)}{16\sin\frac{\pi}{5}}$$

$$=\frac{-\sin\frac{\pi}{5}}{16\sin\frac{\pi}{5}} \qquad \text{(Why?)}$$

$$=-\frac{1}{16}$$

Fill in the blank :

Example 20 If 3 tan (θ – 15°) = tan (θ + 15°), 0° < θ < 90°, then θ = __________

Solution Given that 3 tan (θ – 15°) = tan (θ + 15°) which can be rewritten as

$$\frac{\tan(\theta+15^\circ)}{\tan(\theta-15^\circ)}=\frac{3}{1}.$$

Applying componendo and Dividendo; we get $\frac{\tan(\theta+15^\circ)+\tan(\theta-15^\circ)}{\tan(\theta+15^\circ)-\tan(\theta-15^\circ)}=2$

$$\Rightarrow \frac{\sin(\theta+15^\circ)\cos(\theta-15^\circ)+\sin(\theta-15^\circ)\cos(\theta+15^\circ)}{\sin(\theta+15^\circ)\cos(\theta-15^\circ)-\sin(\theta-15^\circ)\cos(\theta+15^\circ)}=2$$

$$\Rightarrow \frac{\sin 2\theta}{\sin 30^\circ}=2 \quad \text{i.e.,} \quad \sin 2\theta = 1 \qquad \text{(Why?)}$$

giving $\theta=\frac{\pi}{4}$

State whether the following statement is True or False. Justify your answer

Example 21 "The inequality $2^{\sin\theta}+2^{\cos\theta}\geq 2^{1-\frac{1}{\sqrt{2}}}$ holds for all real values of θ"

Solution True. Since $2^{\sin\theta}$ and $2^{\cos\theta}$ are positive real numbers, so A.M. (Arithmetic Mean) of these two numbers is greater or equal to their GM. (Geometric Mean) and hence

$$\frac{2^{\sin\theta}+2^{\cos\theta}}{2} \geq \sqrt{2^{\sin\theta}\times 2^{\cos\theta}} = \sqrt{2^{\sin\theta+\cos\theta}}$$

$$\geq 2^{\frac{\sin\theta+\cos\theta}{2}} = 2^{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta\right)}$$

$$\geq 2^{\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4}+\theta\right)}$$

Since, $-1 \leq \sin\left(\frac{\pi}{4}+\theta\right) \leq 1$, we have

$$\frac{2^{\sin\theta}+2^{\cos\theta}}{2} \geq 2^{\frac{-1}{\sqrt{2}}} \Rightarrow 2^{\sin\theta}+2^{\cos\theta} \geq 2^{1-\frac{1}{\sqrt{2}}}$$

Match each item given under the column C_1 to its correct answer given under column C_2

Example 22

	C_1		C_2
(a)	$\frac{1-\cos x}{\sin x}$	(i)	$\cot^2\frac{x}{2}$
(b)	$\frac{1+\cos x}{1-\cos x}$	(ii)	$\cot\frac{x}{2}$
(c)	$\frac{1+\cos x}{\sin x}$	(iii)	$\lvert\cos x+\sin x\rvert$
(d)	$\sqrt{1+\sin 2x}$	(iv)	$\tan\frac{x}{2}$

Solution

(a) $$\frac{1-\cos x}{\sin x} = \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \tan\frac{x}{2}.$$

Hence (a) matches with (iv) denoted by (a) $\leftrightarrow$ (iv)

(b) $\dfrac{1+\cos x}{1-\cos x} = \dfrac{2\sin^2\frac{x}{2}}{2\sin^2\frac{x}{2}} = \cot^2\frac{x}{2}$. Hence (b) matches with (i) i.e., (b) $\leftrightarrow$ (i)

(c) $\dfrac{1+\cos x}{\sin x} = \dfrac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \cot\frac{x}{2}$.

Hence (c) matches with (ii) i.e., (c) $\leftrightarrow$ (ii)

(d) $\sqrt{1+\sin 2x} = \sqrt{\sin^2 x+\cos^2 x+2\sin x\cos x}$

$= \sqrt{(\sin x+\cos x)^2}$

$= |(\sin x+\cos x)|$. Hence (d) matches with (iii), i.e., (d) $\leftrightarrow$ (iii)

3.3 EXERCISE

Short Answer Type

1. Prove that $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1} = \dfrac{1+\sin A}{\cos A}$

2. If $\dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} = y$, then prove that $\dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}$ is also equal to y.

$\left[\textbf{Hint}:\text{Express } \dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha} = \dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}\cdot\dfrac{1+\cos\alpha+\sin\alpha}{1+\cos\alpha+\sin\alpha}\right]$

3. If $m\sin\theta = n\sin(\theta+2\alpha)$, then prove that $\tan(\theta+\alpha)\cot\alpha = \dfrac{m+n}{m-n}$

[**Hint:** Express $\dfrac{\sin(\theta+2\alpha)}{\sin\theta} = \dfrac{m}{n}$ and apply componendo and dividendo]

4. If $\cos(\alpha+\beta) = \dfrac{4}{5}$ and $\sin(\alpha-\beta) = \dfrac{5}{13}$, where α lie between 0 and $\dfrac{\pi}{4}$, find the value of $\tan 2\alpha$ [**Hint:** Express $\tan 2\alpha$ as $\tan(\alpha+\beta+\alpha-\beta)$]

5. If $\tan x = \frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$

6. Prove that $\cos\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 7\theta \sin 8\theta$.

[**Hint:** Express L.H.S. $= \frac{1}{2}[2\cos\theta \cos\frac{\theta}{2} - 2\cos 3\theta \cos\frac{9\theta}{2}]$]

7. If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$, then show that $a^2 + b^2 = m^2 + n^2$

8. Find the value of $\tan 22°30'$.

[**Hint:** Let $\theta = 45°$, use $\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\theta}{1+\cos\theta}$]

9. Prove that $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$.

10. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then prove that $m^2 - n^2 = 4\sin\theta\tan\theta$

[**Hint:** $m + n = 2\tan\theta$, $m - n = 2\sin\theta$, then use $m^2 - n^2 = (m + n)(m - n)$]

11. If $\tan(A + B) = p$, $\tan(A - B) = q$, then show that $\tan 2A = \frac{p+q}{1-pq}$

[**Hint:** Use $2A = (A + B) + (A - B)$]

12. If $\cos\alpha + \cos\beta = 0 = \sin\alpha + \sin\beta$, then prove that $\cos 2\alpha + \cos 2\beta = -2\cos(\alpha + \beta)$.

[**Hint:** $(\cos\alpha + \cos\beta)^2 - (\sin\alpha + \sin\beta)^2 = 0$]

13. If $\frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}$, then show that $\frac{\tan x}{\tan y} = \frac{a}{b}$ [**Hint:** Use Componendo and Dividendo].

14. If $\tan\theta = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, then show that $\sin\alpha + \cos\alpha = \sqrt{2}\cos\theta$.

[**Hint:** Express $\tan\theta = \tan(\alpha - \frac{\pi}{4}) \Rightarrow \theta = \alpha - \frac{\pi}{4}$]

15. If $\sin\theta + \cos\theta = 1$, then find the general value of θ.

16. Find the most general value of θ satisfying the equation $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$.

17. If $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$, then find the general value of θ.

18. If $2\sin^2\theta = 3\cos\theta$, where $0 \le \theta \le 2\pi$, then find the value of θ.

19. If $\sec x \cos 5x + 1 = 0$, where $0 < x \le \frac{\pi}{2}$, then find the value of x.

Long Answer Type

20. If $\sin(\theta + \alpha) = a$ and $\sin(\theta + \beta) = b$, then prove that $\cos 2(\alpha - \beta) - 4ab\cos(\alpha - \beta) = 1 - 2a^2 - 2b^2$ [**Hint:** Express $\cos(\alpha - \beta) = \cos((\theta + \alpha) - (\theta + \beta))$]

21. If $\cos(\theta + \phi) = m\cos(\theta - \phi)$, then prove that $\tan\theta = \frac{1-m}{1+m}\cot\phi$.

[**Hint:** Express $\frac{\cos(\theta+\phi)}{\cos(\theta-\phi)} = \frac{m}{1}$ and apply Componendo and Dividendo]

22. Find the value of the expression

$$3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left\{\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$$

23. If $a\cos 2\theta + b\sin 2\theta = c$ has α and β as its roots, then prove that

$$\tan\alpha + \tan\beta = \frac{2b}{a+c}.$$

[**Hint:** Use the identities $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$ and $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$].

24. If $x = \sec\phi - \tan\phi$ and $y = \operatorname{cosec}\phi + \cot\phi$ then show that $xy + x - y + 1 = 0$
[**Hint:** Find $xy + 1$ and then show that $x - y = -(xy + 1)$]

25. If θ lies in the first quadrant and $\cos\theta = \frac{8}{17}$, then find the value of

$\cos(30° + \theta) + \cos(45° - \theta) + \cos(120° - \theta)$.

26. Find the value of the expression $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$

[**Hint:** Simplify the expression to $2\left(\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right)$

$$= 2\left[\left(\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8}\right)^2 - 2\cos^2\frac{\pi}{8}\cos^2\frac{3\pi}{8}\right]$$

27. Find the general solution of the equation $5\cos^2\theta + 7\sin^2\theta - 6 = 0$

28. Find the general solution of the equation
$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

29. Find the general solution of the equation $(\sqrt{3} - 1)\cos\theta + (\sqrt{3} + 1)\sin\theta = 2$

[**Hint**: Put $\sqrt{3} - 1 = r\sin\alpha$, $\sqrt{3} + 1 = r\cos\alpha$ which gives $\tan\alpha = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$
$\Rightarrow \alpha = \frac{\pi}{12}$]

Objective Type Questions

Choose the correct answer from the given four options in the Exercises 30 to 59 (M.C.Q.).

30. If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\sin^2\theta + \operatorname{cosec}^2\theta$ is equal to

(A) 1 (B) 4
(C) 2 (D) None of these

31. If $f(x) = \cos^2 x + \sec^2 x$, then

(A) $f(x) < 1$ (B) $f(x) = 1$
(C) $2 < f(x) < 1$ (D) $f(x) \geq 2$

[**Hint**: A.M $\geq$ G.M.]

32. If $\tan\theta = \frac{1}{2}$ and $\tan\phi = \frac{1}{3}$, then the value of $\theta + \phi$ is

(A) $\frac{\pi}{6}$ (B) π (C) 0 (D) $\frac{\pi}{4}$

33. Which of the following is not correct?

(A) $\sin\theta = -\frac{1}{5}$ (B) $\cos\theta = 1$
(C) $\sec\theta = \frac{1}{2}$ (D) $\tan\theta = 20$

34. The value of $\tan 1° \tan 2° \tan 3° \ldots \tan 89°$ is

(A) 0 (B) 1
(C) $\frac{1}{2}$ (D) Not defined

35. The value of $\dfrac{1-\tan^2 15°}{1+\tan^2 15°}$ is

(A) 1 (B) $\sqrt{3}$ (C) $\dfrac{\sqrt{3}}{2}$ (D) 2

36. The value of cos 1° cos 2° cos 3° ... cos 179° is

(A) $\dfrac{1}{\sqrt{2}}$ (B) 0 (C) 1 (D) –1

37. If $\tan\theta = 3$ and θ lies in third quadrant, then the value of $\sin\theta$ is

(A) $\dfrac{1}{\sqrt{10}}$ (B) $-\dfrac{1}{\sqrt{10}}$ (C) $\dfrac{-3}{\sqrt{10}}$ (D) $\dfrac{3}{\sqrt{10}}$

38. The value of tan 75° – cot 75° is equal to

(A) $2\sqrt{3}$ (B) $2+\sqrt{3}$ (C) $2-\sqrt{3}$ (D) 1

39. Which of the following is correct?

(A) $\sin 1° > \sin 1$ (B) $\sin 1° < \sin 1$

(C) $\sin 1° = \sin 1$ (D) $\sin 1° = \dfrac{\pi}{18°}\sin 1$

[**Hint**: 1 radian $= \dfrac{180°}{\pi} = 57°\,30'$ approx]

40. If $\tan\alpha = \dfrac{m}{m+1}$, $\tan\beta = \dfrac{1}{2m+1}$, then $\alpha+\beta$ is equal to

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{3}$ (C) $\dfrac{\pi}{6}$ (D) $\dfrac{\pi}{4}$

41. The minimum value of $3\cos x + 4\sin x + 8$ is

(A) 5 (B) 9 (C) 7 (D) 3

42. The value of tan 3A – tan 2A – tan A is equal to

(A) tan 3A tan 2A tan A

(B) – tan 3A tan 2A tan A

(C) tan A tan 2A – tan 2A tan 3A – tan 3A tan A

(D) None of these

43. The value of sin (45° + θ) – cos (45° – θ) is

(A) 2 cosθ (B) 2 sinθ (C) 1 (D) 0

44. The value of $\cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)$ is

(A) –1 (B) 0 (C) 1 (D) Not defined

45. cos 2θ cos 2ϕ + sin² (θ – ϕ) – sin² (θ + ϕ) is equal to

(A) sin 2(θ + ϕ) (B) cos 2(θ + ϕ)

(C) sin 2(θ – ϕ) (D) cos 2(θ – ϕ)

[**Hint**: Use $\sin^2 A - \sin^2 B = \sin (A + B) \sin (A - B)$]

46. The value of cos 12° + cos 84° + cos 156° + cos 132° is

(A) $\frac{1}{2}$ (B) 1 (C) $-\frac{1}{2}$ (D) $\frac{1}{8}$

47. If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, then tan (2A + B) is equal to

(A) 1 (B) 2 (C) 3 (D) 4

48. The value of $\sin\frac{\pi}{10}\sin\frac{13\pi}{10}$ is

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) $-\frac{1}{4}$ (D) 1

[**Hint**: Use $\sin 18° = \frac{\sqrt{5}-1}{4}$ and $\cos 36° = \frac{\sqrt{5}+1}{4}$]

49. The value of sin 50° – sin 70° + sin 10° is equal to

(A) 1 (B) 0 (C) $\frac{1}{2}$ (D) 2

50. If sin θ + cos θ = 1, then the value of sin 2θ is equal to

(A) 1 (B) $\frac{1}{2}$ (C) 0 (D) –1

51. If $\alpha + \beta = \frac{\pi}{4}$, then the value of $(1 + \tan \alpha)(1 + \tan \beta)$ is

(A) 1 (B) 2

(C) – 2 (D) Not defined

52. If $\sin \theta = \frac{-4}{5}$ and θ lies in third quadrant then the value of $\cos\frac{\theta}{2}$ is

(A) $\frac{1}{5}$ (B) $-\frac{1}{\sqrt{10}}$ (C) $-\frac{1}{\sqrt{5}}$ (D) $\frac{1}{\sqrt{10}}$

53. Number of solutions of the equation $\tan x + \sec x = 2 \cos x$ lying in the interval $[0, 2\pi]$ is

(A) 0 (B) 1 (C) 2 (D) 3

54. The value of $\sin\frac{\pi}{18} + \sin\frac{\pi}{9} + \sin\frac{2\pi}{9} + \sin\frac{5\pi}{18}$ is given by

(A) $\sin\frac{7\pi}{18} + \sin\frac{4\pi}{9}$ (B) 1

(C) $\cos\frac{\pi}{6} + \cos\frac{3\pi}{7}$ (D) $\cos\frac{\pi}{9} + \sin\frac{\pi}{9}$

55. If A lies in the second quadrant and $3 \tan A + 4 = 0$, then the value of $2 \cot A - 5 \cos A + \sin A$ is equal to

(A) $\frac{-53}{10}$ (B) $\frac{23}{10}$ (C) $\frac{37}{10}$ (D) $\frac{7}{10}$

56. The value of $\cos^2 48° - \sin^2 12°$ is

(A) $\frac{\sqrt{5}+1}{8}$ (B) $\frac{\sqrt{5}-1}{8}$

(C) $\frac{\sqrt{5}+1}{5}$ (D) $\frac{\sqrt{5}+1}{2\sqrt{2}}$

[**Hint**: Use $\cos^2 A - \sin^2 B = \cos(A + B)\cos(A - B)$]

57. If $\tan\alpha = \frac{1}{7}$, $\tan\beta = \frac{1}{3}$, then $\cos 2\alpha$ is equal to

(A) $\sin 2\beta$ (B) $\sin 4\beta$ (C) $\sin 3\beta$ (D) $\cos 2\beta$

58. If $\tan\theta = \frac{a}{b}$, then $b\cos 2\theta + a\sin 2\theta$ is equal to

(A) a (B) b (C) $\frac{a}{b}$ (D) None

59. If for real values of x, $\cos\theta = x + \frac{1}{x}$, then

(A) θ is an acute angle (B) θ is right angle

(C) θ is an obtuse angle (D) No value of θ is possible

Fill in the blanks in Exercises 60 to 67 :

60. The value of $\frac{\sin 50°}{\sin 130°}$ is _______ .

61. If $k = \sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$, then the numerical value of k is _______.

62. If $\tan A = \frac{1-\cos B}{\sin B}$, then $\tan 2A =$ _______.

63. If $\sin x + \cos x = a$, then

(i) $\sin^6 x + \cos^6 x =$ _______

(ii) $|\sin x - \cos x| =$ _______.

64. In a triangle ABC with $\angle C = 90°$ the equation whose roots are tan A and tan B is _______.

[**Hint:** $A + B = 90° \Rightarrow \tan A \tan B = 1$ and $\tan A + \tan B = \frac{2}{\sin 2A}$]

65. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) =$ _______.

66. Given $x > 0$, the values of $f(x) = -3\cos\sqrt{3 + x + x^2}$ lie in the interval _______.

67. The maximum distance of a point on the graph of the function $y = \sqrt{3}\sin x + \cos x$ from x-axis is ________.

In each of the Exercises 68 to 75, state whether the statements is True or False? Also give justification.

68. If $\tan A = \dfrac{1 - \cos B}{\sin B}$, then $\tan 2A = \tan B$

69. The equality $\sin A + \sin 2A + \sin 3A = 3$ holds for some real value of A.

70. $\sin 10°$ is greater than $\cos 10°$.

71. $\cos \dfrac{2\pi}{15} \cos \dfrac{4\pi}{15} \cos \dfrac{8\pi}{15} \cos \dfrac{16\pi}{15} = \dfrac{1}{16}$

72. One value of θ which satisfies the equation $\sin^4\theta - 2\sin^2\theta - 1$ lies between 0 and 2π.

73. If $\operatorname{cosec} x = 1 + \cot x$ then $x = 2n\pi,\ 2n\pi + \dfrac{\pi}{2}$

74. If $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$, then $\theta = \dfrac{n\pi}{3} + \dfrac{\pi}{9}$

75. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then $\cos\left(\theta - \dfrac{\pi}{4}\right) = \pm\dfrac{1}{2\sqrt{2}}$

76. In the following match each item given under the column C_1 to its correct answer given under the column C_2 :

(a) $\sin(x + y)\sin(x - y)$	(i) $\cos^2 x - \sin^2 y$
(b) $\cos(x + y)\cos(x - y)$	(ii) $\dfrac{1 - \tan\theta}{1 + \tan\theta}$
(c) $\cot\left(\dfrac{\pi}{4} + \theta\right)$	(iii) $\dfrac{1 + \tan\theta}{1 - \tan\theta}$
(d) $\tan\left(\dfrac{\pi}{4} + \theta\right)$	(iv) $\sin^2 x - \sin^2 y$

Chapter 4

PRINCIPLE OF MATHEMATICAL INDUCTION

4.1 Overview

Mathematical induction is one of the techniques which can be used to prove variety of mathematical statements which are formulated in terms of n, where n is a positive integer.

4.1.1 *The principle of mathematical induction*

Let P(n) be a given statement involving the natural number n such that

(i) The statement is true for $n = 1$, i.e., P(1) is true (or true for any fixed natural number) and

(ii) If the statement is true for $n = k$ (where k is a particular but arbitrary natural number), then the statement is also true for $n = k + 1$, i.e, truth of P(k) implies the truth of P(k + 1). Then P(n) is true for all natural numbers n.

4.2 Solved Examples

Short Answer Type

Prove statements in Examples 1 to 5, by using the Principle of Mathematical Induction for all $n \in \mathbf{N}$, that :

Example 1 $1 + 3 + 5 + ... + (2n - 1) = n^2$

Solution Let the given statement P(n) be defined as P(n) : $1 + 3 + 5 + ... + (2n - 1) = n^2$, for $n \in \mathbf{N}$. Note that P(1) is true, since

$$P(1) : 1 = 1^2$$

Assume that P(k) is true for some $k \in \mathbf{N}$, i.e.,

$$P(k) : 1 + 3 + 5 + ... + (2k - 1) = k^2$$

Now, to prove that P(k + 1) is true, we have

$$1 + 3 + 5 + ... + (2k - 1) + (2k + 1)$$
$$= k^2 + (2k + 1) \qquad \text{(Why?)}$$
$$= k^2 + 2k + 1 = (k + 1)^2$$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all $n \in$ **N**.

Example 2 $\sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}$, for all natural numbers $n \geq 2$.

Solution Let the given statement P(n), be given as

$$P(n): \sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}, \text{ for all natural numbers } n \geq 2.$$

We observe that

$$P(2): \sum_{t=1}^{2-1} t(t+1) = \sum_{t=1}^{1} t(t+1) = 1.2 = \frac{1.2.3}{3}$$

$$= \frac{2.(2-1)(2+1)}{3}$$

Thus, P(n) in true for n = 2.

Assume that P(n) is true for $n = k \in$ **N**.

i.e.,
$$P(k): \sum_{t=1}^{k-1} t(t+1) = \frac{k(k-1)(k+1)}{3}$$

To prove that P(k + 1) is true, we have

$$\sum_{t=1}^{(k+1-1)} t(t+1) = \sum_{t=1}^{k} t(t+1)$$

$$= \sum_{t=1}^{k-1} t(t+1) + k(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1)$$

$$= k(k+1)\left[\frac{k-1+3}{3}\right] = \frac{k(k+1)(k+2)}{3}$$

$$= \frac{(k+1)((k+1)-1))((k+1)+1)}{3}$$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers $n \geq 2$.

Example 3 $\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$, for all natural numbers, $n \geq 2$.

Solution Let the given statement be P(n), i.e.,

$$P(n):\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}, \text{ for all natural numbers, } n\geq 2$$

We, observe that P (2) is true, since

$$\left(1-\frac{1}{2^2}\right)=1-\frac{1}{4} = \frac{4-1}{4}=\frac{3}{4}=\frac{2+1}{2\times 2}$$

Assume that P(n) is true for some $k \in$ **N**, i.e.,

$$P(k):\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$$

Now, to prove that P (k + 1) is true, we have

$$\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{k^2}\right).\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) = \frac{k^2+2k}{2k\,(k+1)} = \frac{(k+1)+1}{2(k+1)}$$

Thus, P (k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers, $n \geq 2$.

Example 4 $2^{2n} - 1$ is divisible by 3.

Solution Let the statement P(n) given as

P(n) : $2^{2n} - 1$ is divisible by 3, for every natural number n.

We observe that P(1) is true, since

$$2^2 - 1 = 4 - 1 = 3.1 \text{ is divisible by } 3.$$

Assume that P(n) is true for some natural number k, i.e.,

P(k): $2^{2k} - 1$ is divisible by 3, i.e., $2^{2k} - 1 = 3q$, where $q \in$ **N**

Now, to prove that P(k + 1) is true, we have

$$P(k+1): 2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k}.2^2 - 1$$
$$= 2^{2k}.4 - 1 = 3.2^{2k} + (2^{2k} - 1)$$

$$= 3.2^{2k} + 3q$$
$$= 3\,(2^{2k} + q) = 3m, \text{ where } m \in \mathbf{N}$$

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers n.

Example 5 $2n + 1 < 2^n$, for all natual numbers $n \geq 3$.

Solution Let P(n) be the given statement, i.e., P(n) : $(2n + 1) < 2^n$ for all natural numbers, $n \geq 3$. We observe that P(3) is true, since

$$2.3 + 1 = 7 < 8 = 2^3$$

Assume that P(n) is true for some natural number k, i.e., $2k + 1 < 2^k$

To prove P(k + 1) is true, we have to show that $2(k + 1) + 1 < 2^{k+1}$. Now, we have

$$2(k + 1) + 1 = 2\,k + 3$$
$$= 2k + 1 + 2 < 2^k + 2 < 2^k \,.\, 2 = 2^{k+1}.$$

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers, $n \geq 3$.

Long Answer Type

Example 6 Define the sequence a_1, a_2, a_3... as follows :

$a_1 = 2$, $a_n = 5\,a_{n-1}$, for all natural numbers $n \geq 2$.

(i) Write the first four terms of the sequence.

(ii) Use the Principle of Mathematical Induction to show that the terms of the sequence satisfy the formula $a_n = 2.5^{n-1}$ for all natural numbers.

Solution

(i) We have $a_1 = 2$

$a_2 = 5a_{2-1} = 5a_1 = 5.2 = 10$

$a_3 = 5a_{3-1} = 5a_2 = 5.10 = 50$

$a_4 = 5a_{4-1} = 5a_3 = 5.50 = 250$

(ii) Let P(n) be the statement, i.e.,

P(n) : $a_n = 2.5^{\,n-1}$ for all natural numbers. We observe that P(1) is true

Assume that P(n) is true for some natural number k, i.e., P(k) : $a_k = 2.5^{k-1}$.

Now to prove that P (k + 1) is true, we have

$$\begin{aligned} P(k+1) : a_{k+1} &= 5.a_k = 5 \cdot (2.5^{k-1}) \\ &= 2.5^k = 2.5^{(k+1)-1} \end{aligned}$$

Thus P(k + 1) is true whenever P (k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers.

Example 7 The distributive law from algebra says that for all real numbers c, a_1 and a_2, we have $c(a_1 + a_2) = ca_1 + ca_2$.

Use this law and mathematical induction to prove that, for all natural numbers, $n \geq 2$, if $c, a_1, a_2, \ldots, a_n$ are any real numbers, then

$$c(a_1 + a_2 + \ldots + a_n) = ca_1 + ca_2 + \ldots + ca_n$$

Solution Let P(n) be the given statement, i.e.,

P(n) : $c(a_1 + a_2 + \ldots + a_n) = ca_1 + ca_2 + \ldots ca_n$ for all natural numbers $n \geq 2$, for $c, a_1, a_2, \ldots a_n \in \mathbf{R}$.

We observe that P(2) is true since

$$c(a_1 + a_2) = ca_1 + ca_2 \qquad \text{(by distributive law)}$$

Assume that P(n) is true for some natural number k, where $k > 2$, i.e.,

$$P(k) : c(a_1 + a_2 + \ldots + a_k) = ca_1 + ca_2 + \ldots + ca_k$$

Now to prove P(k + 1) is true, we have

$$\begin{aligned} P(k+1) : c(a_1 + a_2 + \ldots + a_k + a_{k+1}) \\ &= c((a_1 + a_2 + \ldots + a_k) + a_{k+1}) \\ &= c(a_1 + a_2 + \ldots + a_k) + ca_{k+1} \qquad \text{(by distributive law)} \\ &= ca_1 + ca_2 + \ldots + ca_k + ca_{k+1} \end{aligned}$$

Thus P(k + 1) is true, whenever P (k) is true.

Hence, by the principle of Mathematical Induction, P(n) is true for all natural numbers $n \geq 2$.

Example 8 Prove by induction that for all natural number n

$\sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots + \sin(\alpha + (n-1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Solution Consider P (n) : $\sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots + \sin(\alpha + (n-1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}, \text{ for all natural number } n.$$

We observe that
P (1) is true, since

$$P(1) : \sin\alpha = \frac{\sin(\alpha + 0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

Assume that P(n) is true for some natural numbers k, i.e.,
P (k) : $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (k - 1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Now, to prove that P (k + 1) is true, we have
P (k + 1) : $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (k - 1)\beta) + \sin(\alpha + k\beta)$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} + \sin(\alpha + k\beta)$$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\frac{k\beta}{2} + \sin(\alpha + k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) + \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{k\beta + \beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\,(k+1)\left(\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Thus P $(k + 1)$ is true whenever P (k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural number n.

Example 9 Prove by the Principle of Mathematical Induction that

$1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! - 1$ for all natural numbers n.

Solution Let P(n) be the given statement, that is,

P(n) : $1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! - 1$ for all natural numbers n.

Note that P (1) is true, since

$$P(1) : 1 \times 1! = 1 = 2 - 1 = 2! - 1.$$

Assume that P(n) is true for some natural number k, i.e.,

P(k) : $1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! = (k + 1)! - 1$

To prove P $(k + 1)$ is true, we have

$$\begin{aligned} P(k+1) : 1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! + (k+1) \times (k+1)! \\ &= (k+1)! - 1 + (k+1)! \times (k+1) \\ &= (k+1+1)\ (k+1)! - 1 \\ &= (k+2)\ (k+1)! - 1 = ((k+2)! - 1 \end{aligned}$$

Thus P $(k + 1)$ is true, whenever P (k) is true. Therefore, by the Principle of Mathematical Induction, P (n) is true for all natural number n.

Example 10 Show by the Principle of Mathematical Induction that the sum S_n of the n term of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2$ ... is given by

$$S_n = \begin{cases} \dfrac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \dfrac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Solution Here $P(n) : S_n = \begin{cases} \dfrac{n(n+1)^2}{2}, & \text{when } n \text{ is even} \\ \dfrac{n^2(n+1)}{2}, & \text{when } n \text{ is odd} \end{cases}$

Also, note that any term T_n of the series is given by

$$T_n = \begin{cases} n^2 & \text{if } n \text{ is odd} \\ 2n^2 & \text{if } n \text{ is even} \end{cases}$$

We observe that P(1) is true since

$$P(1) : S_1 = 1^2 = 1 = \frac{1.2}{2} = \frac{1^2.(1+1)}{2}$$

Assume that $P(k)$ is true for some natural number k, i.e.

Case 1 When k is odd, then $k+1$ is even. We have

$P(k+1) : S_{k+1} = 1^2 + 2 \times 2^2 + ... + k^2 + 2 \times (k+1)^2$

$$= \frac{k^2(k+1)}{2} + 2 \times (k+1)^2$$

$$= \frac{(k+1)}{2} [k^2 + 4(k+1)] \text{ (as } k \text{ is odd, } 1^2 + 2 \times 2^2 + ... + k^2 = k^2 \frac{(k+1)}{2}\text{)}$$

$$= \frac{k+1}{2}[k^2 + 4k + 4]$$

$$= \frac{k+1}{2}(k+2)^2 = (k+1) \frac{[(k+1)+1]^2}{2}$$

So $P(k+1)$ is true, whenever $P(k)$ is true in the case when k is odd.

Case 2 When k is even, then $k+1$ is odd.

Now, P $(k + 1) : 1^2 + 2 \times 2^2 + ... + 2.k^2 + (k + 1)^2$

$$= \frac{k(k+1)^2}{2} + (k+1)^2 \quad \left(\text{as } k \text{ is even, } 1^2 + 2 \times 2^2 + ... + 2k^2 = k\frac{(k+1)^2}{2}\right)$$

$$= \frac{(k+1)^2(k+2)}{2} = \frac{(k+1)^2((k+1)+1)}{2}$$

Therefore, P $(k + 1)$ is true, whenever P (k) is true for the case when k is even. Thus P $(k + 1)$ is true whenever P (k) is true for any natural numbers k. Hence, P (n) true for all natural numbers.

Objective Type Questions

Choose the correct answer in Examples 11 and 12 (M.C.Q.)

Example 11 Let P(n) : "$2^n < (1 \times 2 \times 3 \times ... \times n)$". Then the smallest positive integer for which P (n) is true is

(A) 1 (B) 2 (C) 3 (D) 4

Solution Answer is D, since

P (1) : $2 < 1$ is false

P (2) : $2^2 < 1 \times 2$ is false

P (3) : $2^3 < 1 \times 2 \times 3$ is false

But P (4) : $2^4 < 1 \times 2 \times 3 \times 4$ is true

Example 12 A student was asked to prove a statement P (n) by induction. He proved that P $(k + 1)$ is true whenever P (k) is true for all $k > 5 \in$ **N** and also that P (5) is true. On the basis of this he could conclude that P (n) is true

(A) for all $n \in$ **N** (B) for all $n > 5$

(C) for all $n \geq 5$ (D) for all $n < 5$

Solution Answer is (C), since P(5) is true and P$(k + 1)$ is true, whenever P (k) is true.

Fill in the blanks in Example 13 and 14.

Example 13 If P (n) : "$2.4^{2n+1} + 3^{3n+1}$ is divisible by λ for all $n \in$ **N**" is true, then the value of λ is ____

Solution Now, for $n = 1$,

$2.4^{2+1} + 3^{3+1} = 2.4^3 + 3^4 = 2.64 + 81 = 128 + 81 = 209,$

for $n = 2$, $2.4^5 + 3^7 = 8.256 + 2187 = 2048 + 2187 = 4235$

Note that the H.C.F. of 209 and 4235 is 11. So $2.4^{2n+1} + 3^{3n+1}$ is divisible by 11. Hence, λ is 11

Example 14 If P (n) : "$49^n + 16^n + k$ is divisible by 64 for $n \in \mathbf{N}$" is true, then the least negative integral value of k is _______.

Solution For $n = 1$, P(1) : $65 + k$ is divisible by 64.

Thus k, should be -1 since, $65 - 1 = 64$ is divisible by 64.

Example 15 State whether the following proof (by mathematical induction) is true or false for the statement.

$$P(n): 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof By the Principle of Mathematical induction, P(n) is true for $n = 1$,

$1^2 = 1 = \dfrac{1(1+1)(2\cdot1+1)}{6}$. Again for some $k \geq 1$, $k^2 = \dfrac{k(k+1)(2k+1)}{6}$. Now we prove that

$$(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Solution False

Since in the inductive step both the inductive hypothesis and what is to be proved are wrong.

4.3 EXERCISE

Short Answer Type

1. Give an example of a statement P(n) which is true for all $n \geq 4$ but P(1), P(2) and P(3) are not true. Justify your answer.
2. Give an example of a statement P(n) which is true for all n. Justify your answer.

   Prove each of the statements in Exercises 3 - 16 by the Principle of Mathematical Induction :

3. $4^n - 1$ is divisible by 3, for each natural number n.
4. $2^{3n} - 1$ is divisible by 7, for all natural numbers n.
5. $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n.
6. $3^{2n} - 1$ is divisible by 8, for all natural numbers n.

7. For any natural number n, $7^n - 2^n$ is divisible by 5.

8. For any natural number n, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.

9. $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

10. $n(n^2 + 5)$ is divisible by 6, for each natural number n.

11. $n^2 < 2^n$ for all natural numbers $n \geq 5$.

12. $2n < (n + 2)!$ for all natural number n.

13. $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.

14. $2 + 4 + 6 + \ldots + 2n = n^2 + n$ for all natural numbers n.

15. $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all natural numbers n.

16. $1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$ for all natural numbers n.

Long Answer Type

Use the Principle of Mathematical Induction in the following Exercises.

17. A sequence $a_1, a_2, a_3 \ldots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3.7^{n-1}$ for all natural numbers.

18. A sequence $b_0, b_1, b_2 \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all natural numbers k. Show that $b_n = 5 + 4n$ for all natural number n using mathematical induction.

19. A sequence $d_1, d_2, d_3 \ldots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all natural numbers, $k \geq 2$. Show that $d_n = \frac{2}{n!}$ for all $n \in \mathbf{N}$.

20. Prove that for all $n \in \mathbf{N}$

$$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + (n-1)\beta)$$

$$= \frac{\cos\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

21. Prove that, $\cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}$, for all $n \in \mathbf{N}$.

22. Prove that, $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}}$, for all $n \in \mathbf{N}$.

23. Show that $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is a natural number for all $n \in \mathbf{N}$.

24. Prove that $\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}>\frac{13}{24}$, for all natural numbers $n > 1$.

25. Prove that number of subsets of a set containing n distinct elements is 2^n, for all $n \in \mathbf{N}$.

Objective Type Questions

Choose the correct answers in Exercises 26 to 30 (M.C.Q.).

26. If $10^n + 3.4^{n+2} + k$ is divisible by 9 for all $n \in \mathbf{N}$, then the least positive integral value of k is

(A) 5 (B) 3 (C) 7 (D) 1

27. For all $n \in \mathbf{N}$, $3.5^{2n+1} + 2^{3n+1}$ is divisible by

(A) 19 (B) 17 (C) 23 (D) 25

28. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is

(A) 1 (B) 2 (C) 3 (D) 4

Fill in the blanks in the following :

29. If P(n) : $2n < n!$, $n \in \mathbf{N}$, then P(n) is true for all $n \geq$ ________.

State whether the following statement is true or false. Justify.

30. Let P(n) be a statement and let P(k) $\Rightarrow$ P(k + 1), for some natural number k, then P(n) is true for all $n \in \mathbf{N}$.

Chapter 5

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

5.1 Overview

We know that the square of a real number is always non-negative e.g. $(4)^2 = 16$ and $(-4)^2 = 16$. Therefore, square root of 16 is ± 4. What about the square root of a negative number? It is clear that a negative number can not have a real square root. So we need to extend the system of real numbers to a system in which we can find out the square roots of negative numbers. Euler (1707 - 1783) was the first mathematician to introduce the symbol i (iota) for positive square root of -1 i.e., $i = \sqrt{-1}$.

5.1.1 *Imaginary numbers*

Square root of a negative number is called an imaginary number., for example,

$$\sqrt{-9} = \sqrt{-1}\sqrt{9} = i3, \quad \sqrt{-7} = \sqrt{-1}\sqrt{7} = i\sqrt{7}$$

5.1.2 *Integral powers of i*

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = i^2\, i = -i$, $i^4 = (i^2)^2 = (-1)^2 = 1$.

To compute i^n for $n > 4$, we divide n by 4 and write it in the form $n = 4m + r$, where m is quotient and r is remainder $(0 \le r \le 4)$

Hence $\qquad i^n = i^{4m+r} = (i^4)^m \,.\, (i)^r = (1)^m\,(i)^r = i^r$

For example, $\qquad (i)^{39} = i^{4\times 9+3} = (i^4)^9 \,.\, (i)^3 = i^3 = -i$

and $\qquad (i)^{-435} = i^{-(4\times 108+3)} = (i)^{-(4\times 108)} \,.\, (i)^{-3}$

$$= \frac{1}{(i^4)^{108}} \cdot \frac{1}{(i)^3} = \frac{i}{(i)^4} = i$$

(i) If a and b are positive real numbers, then

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{-1}\sqrt{a} \times \sqrt{-1}\sqrt{b} = i\sqrt{a} \times i\sqrt{b} = -\sqrt{ab}$$

(ii) $\sqrt{a}\,.\,\sqrt{b} = \sqrt{ab}$ if a and b are positive or at least one of them is negative or zero. However, $\sqrt{a}\sqrt{b} \ne \sqrt{ab}$ if a and b, both are negative.

5.1.3 *Complex numbers*

(a) A number which can be written in the form $a + ib$, where a, b are real numbers and $i = \sqrt{-1}$ is called a complex number.

(b) If $z = a + ib$ is the complex number, then a and b are called real and imaginary parts, respectively, of the complex number and written as $\mathrm{R}e\,(z) = a$, $\mathrm{I}m\,(z) = b$.

(c) Order relations "greater than" and "less than" are not defined for complex numbers.

(d) If the imaginary part of a complex number is zero, then the complex number is known as purely real number and if real part is zero, then it is called purely imaginary number, for example, 2 is a purely real number because its imaginary part is zero and $3i$ is a purely imaginary number because its real part is zero.

5.1.4 *Algebra of complex numbers*

(a) Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are said to be equal if $a = c$ and $b = d$.

(b) Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers then $z_1 + z_2 = (a + c) + i\,(b + d)$.

5.1.5 *Addition of complex numbers satisfies the following properties*

1. As the sum of two complex numbers is again a complex number, the set of complex numbers is closed with respect to addition.
2. Addition of complex numbers is commutative, i.e., $z_1 + z_2 = z_2 + z_1$
3. Addition of complex numbers is associative, i.e., $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
4. For any complex number $z = x + i\,y$, there exist 0, i.e., $(0 + 0i)$ complex number such that $z + 0 = 0 + z = z$, known as identity element for addition.
5. For any complex number $z = x + iy$, there always exists a number $-z = -a - ib$ such that $z + (-z) = (-z) + z = 0$ and is known as the additive inverse of z.

5.1.6 *Multiplication of complex numbers*

Let $z_1 = a + ib$ and $z_2 = c + id$, be two complex numbers. Then

$z_1 \,.\, z_2 = (a + ib)\,(c + id) = (ac - bd) + i\,(ad + bc)$

1. As the product of two complex numbers is a complex number, the set of complex numbers is closed with respect to multiplication.
2. Multiplication of complex numbers is commutative, i.e., $z_1.z_2 = z_2.z_1$
3. Multiplication of complex numbers is associative, i.e., $(z_1.z_2)\,.\,z_3 = z_1\,.\,(z_2.z_3)$

4. For any complex number $z = x + iy$, there exists a complex number 1, i.e., $(1 + 0i)$ such that
   $z \cdot 1 = 1 \cdot z = z$, known as identity element for multiplication.
5. For any non zero complex number $z = x + i\,y$, there exists a complex number $\frac{1}{z}$ such that $z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1$, i.e., multiplicative inverse of $a + ib = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$.
6. For any three complex numbers z_1, z_2 and z_3,
   $$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$
   and $$(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$$
   i.e., for complex numbers multiplication is distributive over addition.

5.1.7 Let $z_1 = a + ib$ and $z_2(\neq 0) = c + id$. Then

$$z_1 \div z_2 = \frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

5.1.8 *Conjugate of a complex number*

Let $z = a + ib$ be a complex number. Then a complex number obtained by changing the sign of imaginary part of the complex number is called the conjugate of z and it is denoted by $\bar{z}$, i.e., $\bar{z} = a - ib$.

Note that additive inverse of z is $-a - ib$ but conjugate of z is $a - ib$.

We have :

1. $\overline{(\bar{z})} = z$
2. $z + \bar{z} = 2\,\text{Re}\,(z)$, $z - \bar{z} = 2\,i\,\text{Im}(z)$
3. $z = \bar{z}$, if z is purely real.
4. $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
5. $z \cdot \bar{z} = \{\text{Re}\,(z)\}^2 + \{\text{Im}\,(z)\}^2$.
6. $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$, $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$
7. $\overline{(z_1 \cdot z_2)} = (\bar{z}_1)\,(\bar{z}_2)$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\bar{z}_1)}{(\bar{z}_2)}$ $(\bar{z}_2 \neq 0)$

5.1.9 *Modulus of a complex number*

Let $z = a + ib$ be a complex number. Then the positive square root of the sum of square of real part and square of imaginary part is called modulus (absolute value) of z and it is denoted by $|z|$ i.e., $|z| = \sqrt{a^2+b^2}$

In the set of complex numbers $z_1 > z_2$ or $z_1 < z_2$ are meaningless but

$$|z_1| > |z_2| \text{ or } |z_1| < |z_2|$$

are meaningful because $|z_1|$ and $|z_2|$ are real numbers.

5.1.10 *Properties of modulus of a complex number*

1. $|z| = 0 \Leftrightarrow z = 0$ i.e., Re $(z) = 0$ and Im $(z) = 0$
2. $|z| = |\bar{z}| = |-z|$
3. $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$
4. $z\bar{z} = |z|^2$, $|z^2| = |\bar{z}|^2$
5. $|z_1 z_2| = |z_1|.|z_2|$, $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$
6. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$
7. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$
8. $|z_1 + z_2| \leq |z_1| + |z_2|$
9. $|z_1 - z_2| \geq |z_1| - |z_2|$
10. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

    In particular:

    $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
11. As stated earlier multiplicative inverse (reciprocal) of a complex number $z = a + ib$ $(\neq 0)$ is

$$\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

5.2 Argand Plane

A complex number $z = a + ib$ can be represented by a unique point P (a, b) in the cartesian plane referred to a pair of rectangular axes. The complex number $0 + 0i$ represent the origin 0 (0, 0). A purely real number a, i.e., $(a + 0i)$ is represented by the point $(a, 0)$ on x - axis. Therefore, x-axis is called real axis. A purely imaginary number

ib, i.e., $(0 + ib)$ is represented by the point $(0, b)$ on y-axis. Therefore, y-axis is called imaginary axis.

Similarly, the representation of complex numbers as points in the plane is known as **Argand diagram**. The plane representing complex numbers as points is called complex plane or Argand plane or Gaussian plane.

If two complex numbers z_1 and z_2 be represented by the points P and Q in the complex plane, then

$$|z_1 - z_2| = \text{PQ}$$

5.2.1 *Polar form of a complex number*

Let P be a point representing a non-zero complex number $z = a + ib$ in the Argand plane. If OP makes an angle θ with the positive direction of x-axis, then $z = r(\cos\theta + i\sin\theta)$ is called the polar form of the complex number, where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$. Here θ is called argument or amplitude of z and we write it as $\arg(z) = \theta$.

The unique value of θ such that $-\pi \le \theta \le \pi$ is called the principal argument.

$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

5.2.2 *Solution of a quadratic equation*

The equations $ax^2 + bx + c = 0$, where a, b and c are numbers (real or complex, $a \neq 0$) is called the general quadratic equation in variable x. The values of the variable satisfying the given equation are called roots of the equation.

The quadratic equation $ax^2 + bx + c = 0$ with real coefficients has two roots given by $\frac{-b+\sqrt{D}}{2a}$ and $\frac{-b-\sqrt{D}}{2a}$, where $D = b^2 - 4ac$, called the discriminant of the equation.

Notes

1. When $D = 0$, roots of the quadratic equation are real and equal. When $D > 0$, roots are real and unequal.
   Further, if $a, b, c \in \mathbf{Q}$ and D is a perfect square, then the roots of the equation are rational and unequal, and if $a, b, c \in \mathbf{Q}$ and D is not a perfect square, then the roots are irrational and occur in pair.

When D < 0, roots of the quadratic equation are non real (or complex).

2. Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$, then sum of the roots

$(\alpha + \beta) = \dfrac{-b}{a}$ and the product of the roots $(\alpha \, . \, \beta) = \dfrac{c}{a}$.

3. Let S and P be the sum of roots and product of roots, respectively, of a quadratic equation. Then the quadratic equation is given by $x^2 - Sx + P = 0$.

5.2 Solved Exmaples

Short Answer Type

Example 1 Evaluate : $(1 + i)^6 + (1 - i)^3$

Solution $(1 + i)^6 = \{(1 + i)^2\}^3 = (1 + i^2 + 2i)^3 = (1 - 1 + 2i)^3 = 8\, i^3 = -8i$

and $\quad (1 - i)^3 = 1 - i^3 - 3i + 3i^2 = 1 + i - 3i - 3 = -2 - 2i$

Therefore, $\quad (1 + i)^6 + (1 - i)^3 = -8i - 2 - 2i = -2 - 10i$

Example 2 If $(x+iy)^{\frac{1}{3}} = a + ib$, where $x, y, a, b \in$ R, show that $\dfrac{x}{a} - \dfrac{y}{b} = -2\,(a^2 + b^2)$

Solution $(x+iy)^{\frac{1}{3}} = a + ib$

$\Rightarrow \quad x + iy = (a + ib)^3$

i.e., $\quad x + iy = a^3 + i^3 b^3 + 3iab\,(a + ib)$

$= a^3 - ib^3 + i3a^2b - 3ab^2$

$= a^3 - 3ab^2 + i\,(3a^2b - b^3)$

$\Rightarrow \quad x = a^3 - 3ab^2$ and $y = 3a^2b - b^3$

Thus $\quad \dfrac{x}{a} = a^2 - 3b^2$ and $\dfrac{y}{b} = 3a^2 - b^2$

So, $\quad \dfrac{x}{a} - \dfrac{y}{b} = a^2 - 3b^2 - 3a^2 + b^2 = -2\,a^2 - 2b^2 = -2\,(a^2 + b^2).$

Example 3 Solve the equation $z^2 = \bar{z}$, where $z = x + iy$

Solution $z^2 = \bar{z} \quad \Rightarrow \quad x^2 - y^2 + i2xy = x - iy$

Therefore, $x^2 - y^2 = x \quad \ldots (1) \quad$ and $\quad 2xy = -y \quad \ldots (2)$

From (2), we have $y = 0$ or $x = -\dfrac{1}{2}$

When $y = 0$, from (1), we get $x^2 - x = 0$, i.e., $x = 0$ or $x = 1$.

When $x = -\dfrac{1}{2}$, from (1), we get $y^2 = \dfrac{1}{4} + \dfrac{1}{2}$ or $y^2 = \dfrac{3}{4}$, i.e., $y = \pm\dfrac{\sqrt{3}}{2}$.

Hence, the solutions of the given equation are

$$0 + i0,\ 1 + i0,\ -\frac{1}{2} + i\,\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Example 4 If the imaginary part of $\dfrac{2z+1}{iz+1}$ is -2, then show that the locus of the point representing z in the argand plane is a straight line.

Solution Let $z = x + iy$. Then

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix}$$

$$= \frac{\{(2x+1)+i2y\}}{\{(1-y)+ix\}} \times \frac{\{(1-y)-ix\}}{\{(1-y)-ix\}}$$

$$= \frac{(2x+1-y)+i(2y-2y^2-2x^2-x)}{1+y^2-2y+x^2}$$

Thus $\quad \text{Im}\left(\dfrac{2z+1}{iz+1}\right) = \dfrac{2y-2y^2-2x^2-x}{1+y^2-2y+x^2}$

But $\quad \text{Im}\left(\dfrac{2z+1}{iz+1}\right) = -2 \quad$ (Given)

So $\quad \dfrac{2y-2y^2-2x^2-x}{1+y^2-2y+x^2} = -2$

$\Rightarrow \quad 2y - 2y^2 - 2x^2 - x = -2 - 2y^2 + 4y - 2x^2$

i.e., $\quad x + 2y - 2 = 0$, which is the equation of a line.

Example 5 If $|z^2 - 1| = |z|^2 + 1$, then show that z lies on imaginary axis.

Solution Let $z = x + iy$. Then $|z^2 - 1| = |z|^2 + 1$

$\Rightarrow \quad |x^2 - y^2 - 1 + i\,2xy| = |x + iy|^2 + 1$

$\Rightarrow \quad (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$

$\Rightarrow \quad 4x^2 = 0 \quad$ i.e., $\quad x = 0$

Hence z lies on y-axis.

Example 6 Let z_1 and z_2 be two complex numbers such that $\bar{z}_1 + i\bar{z}_2 = 0$ and $\arg(z_1 z_2) = \pi$. Then find $\arg(z_1)$.

Solution Given that $\bar{z}_1 + i\bar{z}_2 = 0$

$\Rightarrow \quad z_1 = iz_2$, i.e., $z_2 = -iz_1$

Thus $\quad \arg(z_1 z_2) = \arg z_1 + \arg(-iz_1) = \pi$

$\Rightarrow \quad \arg(-iz_1^2) = \pi$

$\Rightarrow \quad \arg(-i) + \arg(z_1^2) = \pi$

$\Rightarrow \quad \arg(-i) + 2\arg(z_1) = \pi$

$\Rightarrow \quad \dfrac{-\pi}{2} + 2\arg(z_1) = \pi$

$\Rightarrow \quad \arg(z_1) = \dfrac{3\pi}{4}$

Example 7 Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$.

Then show that $\arg(z_1) - \arg(z_2) = 0$.

Solution Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

where $\quad r_1 = |z_1|$, $\arg(z_1) = \theta_1$, $r_2 = |z_2|$, $\arg(z_2) = \theta_2$.

We have, $\quad |z_1 + z_2| = |z_1| + |z_2|$

$= |r_1(\cos\theta_1 + \cos\theta_2) + r_2(\cos\theta_2 + \sin\theta_2)| = r_1 + r_2$

$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) = (r_1 + r_2)^2 \Rightarrow \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \theta_1 - \theta_2$ i.e. $\arg z_1 = \arg z_2$

Example 8 If z_1, z_2, z_3 are complex numbers such that

$|z_1| = |z_2| = |z_3| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$, then find the value of $|z_1 + z_2 + z_3|$.

Solution $|z_1| = |z_2| = |z_3| = 1$

$$\Rightarrow \qquad |z_1|^2 = |z_2|^2 = |z_3|^2 = 1$$

$$\Rightarrow \qquad z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = 1$$

$$\Rightarrow \qquad \bar{z}_1 = \frac{1}{z_1},\ \bar{z}_2 = \frac{1}{z_2},\ \bar{z}_3 = \frac{1}{z_3}$$

Given that $\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$

$$\Rightarrow \qquad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1,\ \text{i.e.,}\ \left|\overline{z_1 + z_2 + z_3}\right| = 1$$

$$\Rightarrow \qquad |z_1 + z_2 + z_3| = 1$$

Example 9 If a complex number z lies in the interior or on the boundary of a circle of radius 3 units and centre (– 4, 0), find the greatest and least values of $|z+1|$.

Solution Distance of the point representing z from the centre of the circle is $|z-(-4+i0)| = |z+4|$.

According to given condition $|z+4| \le 3$.

Now $|z+1| = |z+4-3| \le |z+4| + |-3| \le 3+3=6$

Therefore, greatest value of $|z + 1|$ is 6.

Since least value of the modulus of a complex number is zero, the least value of $|z+1| = 0$.

Example 10 Locate the points for which $3 < |z| < 4$

Solution $|z| < 4 \Rightarrow x^2 + y^2 < 16$ which is the interior of circle with centre at origin and radius 4 units, and $|z| > 3 \Rightarrow x^2 + y^2 > 9$ which is exterior of circle with centre at origin and radius 3 units. Hence $3 < |z| < 4$ is the portion between two circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$.

Example 11 Find the value of $2x^4 + 5x^3 + 7x^2 - x + 41$, when $x = -2 - \sqrt{3}\,i$

Solution $x + 2 = -\sqrt{3}\,i \Rightarrow x^2 + 4x + 7 = 0$

Therefore
$$2x^4 + 5x^3 + 7x^2 - x + 41 = (x^2 + 4x + 7)(2x^2 - 3x + 5) + 6$$
$$= 0 \times (2x^2 - 3x + 5) + 6 = 6.$$

Example 12 Find the value of P such that the difference of the roots of the equation $x^2 - Px + 8 = 0$ is 2.

Solution Let α, β be the roots of the equation $x^2 - Px + 8 = 0$

Therefore $\alpha + \beta = P$ and $\alpha . \beta = 8$.

Now $\alpha - \beta = \pm\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

Therefore $2 = \pm\sqrt{P^2 - 32}$

$\Rightarrow$ $P^2 - 32 = 4$, i.e., $P = \pm 6$.

Example 13 Find the value of a such that the sum of the squares of the roots of the equation $x^2 - (a - 2)x - (a + 1) = 0$ is least.

Solution Let α, β be the roots of the equation

Therefore, $\alpha + \beta = a - 2$ and $\alpha\beta = -(a + 1)$

Now

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$
$$= (a-2)^2 + 2(a + 1)$$
$$= (a - 1)^2 + 5$$

Therefore, $\alpha^2 + \beta^2$ will be minimum if $(a - 1)^2 = 0$, i.e., $a = 1$.

Long Answer Type

Example 14 Find the value of k if for the complex numbers z_1 and z_2,

$$|1-\bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1-|z_1|^2)(1-|z_2|^2)$$

Solution

$$\text{L.H.S.} = |1-\bar{z}_1 z_2|^2 - |z_1 - z_2|^2$$
$$= (1-\bar{z}_1 z_2)\,(\overline{1-\bar{z}_1 z_2}) - (z_1 - z_2)\,(\overline{z_1 - z_2})$$
$$= (1-\bar{z}_1 z_2)\,(1-z_1\bar{z}_2) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$
$$= 1 + z_1\bar{z}_1\, z_2\bar{z}_2 - z_1\bar{z}_1 - z_2\bar{z}_2$$
$$= 1 + |z_1|^2 \cdot |z_2|^2 - |z_1|^2 - |z_2|^2$$
$$= (1-|z_1|^2)\,(1-|z_2|^2)$$

$$\text{R.H.S.} = k\,(1 - |z_1|^2)\,(1-|z_2|^2)$$

$\Rightarrow$ $k = 1$

Hence, equating LHS and RHS, we get $k = 1$.

Example 15 If z_1 and z_2 both satisfy $z + \bar{z} = 2|z-1|$ $\arg(z_1 - z_2) = \frac{\pi}{4}$, then find $Im(z_1 + z_2)$.

Solution Let $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then $\quad z + \bar{z} = 2|z-1|$

$\Rightarrow \quad (x + iy) + (x - iy) = 2\,|x-1+iy|$

$\Rightarrow \quad 2x = 1 + y^2 \quad \ldots (1)$

Since z_1 and z_2 both satisfy (1), we have

$2x_1 = 1 + y_1^2 \ldots$ and $2x_2 = 1 + y_2^2$

$\Rightarrow \quad 2(x_1 - x_2) = (y_1 + y_2)(y_1 - y_2)$

$\Rightarrow \quad 2 = (y_1 + y_2)\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \quad \ldots (2)$

Again $\quad z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

Therefore, $\quad \tan\theta = \frac{y_1 - y_2}{x_1 - x_2}$, where $\theta = \arg(z_1 - z_2)$

$\Rightarrow \quad \tan\frac{\pi}{4} = \frac{y_1 - y_2}{x_1 - x_2} \quad \left(\text{since } \theta = \frac{\pi}{4}\right)$

i.e., $\quad 1 = \frac{y_1 - y_2}{x_1 - x_2}$

From (2), we get $2 = y_1 + y_2$, i.e., $\text{Im}(z_1 + z_2) = 2$

Objective Type Questions

Example 16 Fill in the blanks:

(i) The real value of 'a' for which $3i^3 - 2ai^2 + (1 - a)i + 5$ is real is ________.

(ii) If $|z| = 2$ and $\arg(z) = \frac{\pi}{4}$, then $z =$ ________.

(iii) The locus of z satisfying $\arg(z) = \frac{\pi}{3}$ is _______.

(iv) The value of $(-\sqrt{-1})^{4n-3}$, where $n \in \mathbf{N}$, is ______.

(v) The conjugate of the complex number $\frac{1-i}{1+i}$ is _____.

(vi) If a complex number lies in the third quadrant, then its conjugate lies in the _______.

(vii) If $(2 + i)(2 + 2i)(2 + 3i) \ldots (2 + ni) = x + iy$, then $5.8.13 \ldots (4 + n^2) =$ ______.

Solution

(i) $3i^3 - 2ai^2 + (1 - a)i + 5 = -3i + 2a + 5 + (1 - a)i$

$= 2a + 5 + (-a - 2)\, i$, which is real if $-a - 2 = 0$ i.e. $a = -2$.

(ii) $z = |z|\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\,(1+i)$

(iii) Let $z = x + iy$. Then its polar form is $z = r(\cos\theta + i\sin\theta)$, where $\tan\theta = \frac{y}{x}$ and θ is $\arg(z)$. Given that $\theta = \frac{\pi}{3}$. Thus.

$\tan\frac{\pi}{3} = \frac{y}{x} \Rightarrow y = \sqrt{3}x$, where $x > 0, y > 0$.

Hence, locus of z is the part of $y = \sqrt{3}x$ in the first quadrant except origin.

(iv) Here $(-\sqrt{-1})^{4n-3} = (-i)^{4n-3} = (-i)^{4n}\,(-i)^{-3} = \frac{1}{(-i)^3}$

$$= \frac{1}{-i^3} = \frac{1}{i} = \frac{i}{i^2} = -i$$

(v) $\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1+i^2-2i}{1-i^2} = \frac{1-1-2i}{1+1} = -i$

Hence, conjugate of $\frac{1-i}{1+i}$ is i.

(vi) Conjugate of a complex number is the image of the complex number about the x-axis. Therefore, if a number lies in the third quadrant, then its image lies in the second quadrant.

(vii) Given that $(2 + i)(2 + 2i)(2 + 3i) \ldots (2 + ni) = x + iy$ ... (1)

$\Rightarrow \quad \overline{(2+i)}\;\overline{(2+2i)}\;\overline{(2+3i)} \ldots \overline{(2+ni)} = \left(\overline{x+iy}\right) = (x-iy)$

i.e., $\quad (2 - i)(2 - 2i)(2 - 3i) \ldots (2 - ni) = x - iy$ ... (2)

Multiplying (1) and (2), we get 5.8.13 ... $(4 + n^2) = x^2 + y^2$.

Example 17 State true or false for the following:

(i) Multiplication of a non-zero complex number by i rotates it through a right angle in the anti- clockwise direction.

(ii) The complex number $\cos\theta + i \sin\theta$ can be zero for some θ.

(iii) If a complex number coincides with its conjugate, then the number must lie on imaginary axis.

(iv) The argument of the complex number $z = (1 + i\sqrt{3})(1 + i)(\cos\theta + i \sin\theta)$ is $\frac{7\pi}{12} + \theta$

(v) The points representing the complex number z for which $|z+1| < |z-1|$ lies in the interior of a circle.

(vi) If three complex numbers z_1, z_2 and z_3 are in A.P., then they lie on a circle in the complex plane.

(vii) If n is a positive integer, then the value of $i^n + (i)^{n+1} + (i)^{n+2} + (i)^{n+3}$ is 0.

Solution

(i) True. Let $z = 2 + 3i$ be complex number represented by OP. Then $iz = -3 + 2i$, represented by OQ, where if OP is rotated in the anticlockwise direction through a right angle, it coincides with OQ.

(ii) False. Because $\cos\theta + i\sin\theta = 0 \Rightarrow \cos\theta = 0$ and $\sin\theta = 0$. But there is no value of θ for which $\cos\theta$ and $\sin\theta$ both are zero.

(iii) False, because $x + iy = x - iy \Rightarrow y = 0 \Rightarrow$ number lies on x-axis.

(iv) True, $\arg(z) = \arg(1 + i\sqrt{3}) + \arg(1 + i) + \arg(\cos\theta + i\sin\theta)$

$$\frac{\pi}{3}+\frac{\pi}{4}+\theta=\frac{7\pi}{12}+\theta$$

(v) False, because $|x+iy+1| < |x+iy-1|$

$\Rightarrow \quad (x + 1)^2 + y^2 < (x - 1)^2 + y^2$ which gives $4x < 0$.

(vi) False, because if z_1, z_2 and z_3 are in A.P., then $z_2 = \frac{z_1 + z_3}{2} \Rightarrow z_2$ is the midpoint of z_1 and z_3, which implies that the points z_1, z_2, z_3 are collinear.

(vii) True, because $i^n + (i)^{n+1} + (i)^{n+2} + (i)^{n+3}$

$$= i^n (1 + i + i^2 + i^3) = i^n (1 + i - 1 - i)$$

$$= i^n (0) = 0$$

Example 18 Match the statements of column A and B.

Column A		Column B	
(a)	The value of $1+i^2+i^4+i^6+\ldots i^{20}$ is	(i)	purely imaginary complex number
(b)	The value of i^{-1097} is	(ii)	purely real complex number
(c)	Conjugate of $1+i$ lies in	(iii)	second quadrant
(d)	$\dfrac{1+2i}{1-i}$ lies in	(iv)	Fourth quadrant
(e)	If $a, b, c \in$ R and $b^2 - 4ac < 0$, then the roots of the equation $ax^2 + bx + c = 0$ are non real (complex) and	(v)	may not occur in conjugate pairs
(f)	If $a, b, c \in$ R and $b^2 - 4ac > 0$, and $b^2 - 4ac$ is a perfect square, then the roots of the equation $ax^2 + bx + c = 0$	(vi)	may occur in conjugate pairs

Solution

(a) $\Leftrightarrow$ (ii), because $1 + i^2 + i^4 + i^6 + \ldots + i^{20}$
$= 1 - 1 + 1 - 1 + \ldots + 1 = 1$ (which is purely a real complex number)

(b) $\Leftrightarrow$ (i), because $i^{-1097} = \dfrac{1}{(i)^{1097}} = \dfrac{1}{i^{4\times 274+1}} = \dfrac{1}{\{(i)^4\}^{274}(i)} = \dfrac{1}{i} = \dfrac{i}{i^2} = -i$

which is purely imaginary complex number.

(c) $\Leftrightarrow$ (iv), conjugate of $1 + i$ is $1 - i$, which is represented by the point $(1, -1)$ in the fourth quadrant.

(d) $\Leftrightarrow$ (iii), because $\dfrac{1+2i}{1-i} = \dfrac{1+2i}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{-1+3i}{2} = -\dfrac{1}{2} + \dfrac{3}{2}i$, which is represented by the point $\left(-\dfrac{1}{2}, \dfrac{3}{2}\right)$ in the second quadrant.

(e) $\Leftrightarrow$ (vi), If $b^2 - 4ac < 0 = \text{D} < 0$, i.e., square root of D is a imaginary number, therefore, roots are $x = \dfrac{-b \pm \text{Imaginary Number}}{2a}$, i.e., roots are in conjugate pairs.

(f) $\Leftrightarrow$ (v), Consider the equation $x^2 - (5 + \sqrt{2})\, x + 5\sqrt{2} = 0$, where $a = 1$, $b = -(5 + \sqrt{2})$, $c = 5\sqrt{2}$, clearly $a, b, c \in$ R.

Now $D = b^2 - 4ac = \{-(5 + \sqrt{2})\}^2 - 4.1.5\sqrt{2} = (5 - \sqrt{2})^2$.

Therefore $x = \dfrac{5+\sqrt{2} \pm 5-\sqrt{2}}{2} = 5, \sqrt{2}$ which do not form a conjugate pair.

Example 19 What is the value of $\dfrac{i^{4n+1} - i^{4n-1}}{2}$?

Solution i, because $\dfrac{i^{4n+1} - i^{4n-1}}{2} = \dfrac{i^{4n}i - i^{4n}i^{-i}}{2}$

$$= \frac{i - \frac{1}{i}}{2} = \frac{i^2 - 1}{2i} = \frac{-2}{2i} = i$$

Example 20 What is the smallest positive integer n, for which $(1 + i)^{2n} = (1 - i)^{2n}$?

Solution $n = 2$, because $(1 + i)^{2n} = (1 - i)^{2n} = \left(\dfrac{1+i}{1-i}\right)^{2n} = 1$

$\Rightarrow$ $(i)^{2n} = 1$ which is possible if $n = 2$ $(\therefore\ i^4 = 1)$

Example 21 What is the reciprocal of $3 + \sqrt{7}\, i$

Solution Reciprocal of $z = \dfrac{\bar{z}}{|z|^2}$

Therefore, reciprocal of $3 + \sqrt{7}\ i = \dfrac{3 - \sqrt{7}\, i}{16} = \dfrac{3}{16} - \dfrac{\sqrt{7}\, i}{16}$

Example 22 If $z_1 = \sqrt{3} + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$, then find the quadrant in which $\left(\dfrac{z_1}{z_2}\right)$ lies.

Solution $\dfrac{z_1}{z_2} = \dfrac{\sqrt{3} + i\sqrt{3}}{\sqrt{3} + i} = \left(\dfrac{3+\sqrt{3}}{4}\right) + \left(\dfrac{3-\sqrt{3}}{4}\right)i$

which is represented by a point in first quadrant.

Example 23 What is the conjugate of $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$?

Solution Let

$$z = \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}} \times \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}+\sqrt{5-12i}}$$

$$= \frac{5+12i+5-12i+2\sqrt{25+144}}{5+12i-5+12i}$$

$$= \frac{3}{2i} = \frac{3i}{-2} = 0-\frac{3}{2}i$$

Therefore, the conjugate of $z = 0 + \dfrac{3}{2}i$

Example 24 What is the principal value of amplitude of $1 - i$?

Solution Let θ be the principle value of amplitude of $1 - i$. Since

$$\tan\theta = -1 \Rightarrow \tan\theta = \tan\left(-\frac{\pi}{4}\right) \Rightarrow \theta = -\frac{\pi}{4}$$

Example 25 What is the polar form of the complex number $(i^{25})^3$?

Solution $z = (i^{25})^3 = (i)^{75} = i^{4\times 18+3} = (i^4)^{18}\ (i)^3$

$= i^3 = -i = 0 - i$

Polar form of $z = r(\cos\theta + i\sin\theta)$

$$= 1\left\{\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right\}$$

$$= \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

Example 26 What is the locus of z, if amplitude of $z - 2 - 3i$ is $\dfrac{\pi}{4}$?

Solution Let $z = x + iy$. Then $z - 2 - 3i = (x - 2) + i(y - 3)$

Let θ be the amplitude of $z - 2 - 3i$. Then $\tan\theta = \dfrac{y-3}{x-2}$

$$\Rightarrow \qquad \tan\frac{\pi}{4} = \frac{y-3}{x-2}\left(\text{since}\,\theta = \frac{\pi}{4}\right)$$

$\Rightarrow \qquad 1 = \dfrac{y-3}{x-2}$ i.e. $x - y + 1 = 0$

Hence, the locus of z is a straight line.

Example 27 If $1 - i$, is a root of the equation $x^2 + ax + b = 0$, where $a, b \in \mathbf{R}$, then find the values of a and b.

Solution Sum of roots $\dfrac{-a}{1} = (1 - i) + (1 + i) \Rightarrow a = -2$.

(since non real complex roots occur in conjugate pairs)

Product of roots, $\dfrac{b}{1} = (1-i)(1+i) \Rightarrow b = 2$

Choose the correct options out of given four options in each of the Examples from 28 to 33 (M.C.Q.).

Example 28 $1 + i^2 + i^4 + i^6 + ... + i^{2n}$ is

(A) positive (B) negative

(C) 0 (D) can not be evaluated

Solution (D), $1 + i^2 + i^4 + i^6 + ... + i^{2n} = 1 - 1 + 1 - 1 + ... (-1)^n$

which can not be evaluated unless n is known.

Example 29 If the complex number $z = x + iy$ satisfies the condition $|z+1| = 1$, then z lies on

(A) x-axis

(B) circle with centre (1, 0) and radius 1

(C) circle with centre (–1, 0) and radius 1

(D) y-axis

Solution (C), $|z+1| = 1 \Rightarrow |(x+1)+iy| = 1$

$\Rightarrow \qquad (x+1)^2 + y^2 = 1$

which is a circle with centre (–1, 0) and radius 1.

Example 30 The area of the triangle on the complex plane formed by the complex numbers z, $-iz$ and $z + iz$ is:

(A) $|z|^2$ (B) $|\bar{z}|^2$

(C) $\dfrac{|z|^2}{2}$ (D) none of these

Solution (C), Let $z = x + iy$. Then $-iz = y - ix$. Therefore,

$$z + iz = (x - y) + i\,(x + y)$$

Required area of the triangle $= \frac{1}{2}(x^2 + y^2) = \frac{|z|^2}{2}$

Example 31 The equation $|z+1-i| = |z-1+i|$ represents a

(A) straight line (B) circle

(C) parabola (D) hyperbola

Solution (A), $|z+1-i| = |z-1+i|$

$\Rightarrow$ $|z-(-1+i)| = |z-(1-i)|$

$\Rightarrow$ PA = PB, where A denotes the point (–1, 1), B denotes the point (1, –1) and P denotes the point (x, y)

$\Rightarrow$ z lies on the perpendicular bisector of the line joining A and B and perpendicular bisector is a straight line.

Example 32 Number of solutions of the equation $z^2 + |z|^2 = 0$ is

(A) 1 (B) 2

(C) 3 (D) infinitely many

Solution (D), $z^2 + |z|^2 = 0$, $z \neq 0$

$\Rightarrow$ $x^2 - y^2 + i2xy + x^2 + y^2 = 0$

$\Rightarrow$ $2x^2 + i2xy = 0 \Rightarrow 2x\,(x + iy) = 0$

$\Rightarrow$ $x = 0$ or $x + iy = 0$ (not possible)

Therefore, $x = 0$ and $z \neq 0$

So y can have any real value. Hence infinitely many solutions.

Example 33 The amplitude of $\sin\frac{\pi}{5} + i\,(1 - \cos\frac{\pi}{5})$ is

(A) $\frac{2\pi}{5}$ (B) $\frac{\pi}{5}$ (C) $\frac{\pi}{15}$ (D) $\frac{\pi}{10}$

Solution (D), Here $r\cos\theta = \sin\left(\frac{\pi}{5}\right)$ and $r\sin\theta = 1 - \cos\frac{\pi}{5}$

Therefore, $\tan\theta = \dfrac{1-\cos\dfrac{\pi}{5}}{\sin\dfrac{\pi}{5}} = \dfrac{2\sin^2\left(\dfrac{\pi}{10}\right)}{2\sin\left(\dfrac{\pi}{10}\right).\cos\left(\dfrac{\pi}{10}\right)}$

$\Rightarrow \quad \tan\theta = \tan\left(\dfrac{\pi}{10}\right)$ i.e., $\theta = \dfrac{\pi}{10}$

5.3 EXERCISE

Short Answer Type

1. For a positive integer n, find the value of $(1-i)^n\left(1-\dfrac{1}{i}\right)^n$

2. Evaluate $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $n \in \mathbf{N}$.

3. If $\left(\dfrac{1+i}{1-i}\right)^3 - \left(\dfrac{1-i}{1+i}\right)^3 = x + iy$, then find (x, y).

4. If $\dfrac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

5. If $\left(\dfrac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b).

6. If $a = \cos\theta + i\sin\theta$, find the value of $\dfrac{1+a}{1-a}$.

7. If $(1 + i)\, z = (1 - i)\, \bar{z}$, then show that $z = -i\bar{z}$.

8. If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in \mathbf{R}$, represents a circle.

9. If the real part of $\dfrac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

10. Show that the complex number z, satisfying the condition $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$ lies on a circle.

11. Solve the equation $|z| = z + 1 + 2i$.

Long Answer Type

12. If $|z+1| = z + 2(1+i)$, then find z.

13. If $\arg(z-1) = \arg(z+3i)$, then find $x - 1 : y$. where $z = x + iy$

14. Show that $\left|\frac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.

15. If $\frac{z-1}{z+1}$ is a purely imaginary number $(z \neq -1)$, then find the value of $|z|$.

16. z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.

17. If $|z_1| = 1$ $(z_1 \neq -1)$ and $z_2 = \frac{z_1 - 1}{z_1 + 1}$, then show that the real part of z_2 is zero.

18. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right).$$

19. If $|z_1| = |z_2| = \ldots = |z_n| = 1$, then

show that $|z_1 + z_2 + z_3 + \ldots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \ldots + \frac{1}{z_n}\right|$.

20. If for complex numbers z_1 and z_2, $\arg(z_1) - \arg(z_2) = 0$, then show that $|z_1 - z_2| = |z_1| - |z_2|$

21. Solve the system of equations $\text{Re}(z^2) = 0$, $|z| = 2$.

22. Find the complex number satisfying the equation $z + \sqrt{2}\,|(z+1)| + i = 0$.

23. Write the complex number $z = \frac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ in polar form.

24. If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then show that $\bar{z}w = -i$.

Objective Type Questions

25. Fill in the blanks of the following

(i) For any two complex numbers z_1, z_2 and any real numbers a, b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 =$ .....

(ii) The value of $\sqrt{-25} \times \sqrt{-9}$ is ....

(iii) The number $\dfrac{(1-i)^3}{1-i^3}$ is equal to ....

(iv) The sum of the series $i + i^2 + i^3 + \ldots$ upto 1000 terms is ....

(v) Multiplicative inverse of $1 + i$ is ....

(vi) If z_1 and z_2 are complex numbers such that $z_1 + z_2$ is a real number, then $z_2 =$ ....

(vii) $\arg(z) + \arg \bar{z}$ $(\bar{z} \neq 0)$ is ....

(viii) If $|z+4| \leq 3$, then the greatest and least values of $|z+1|$ are ..... and .....

(ix) If $\left|\dfrac{z-2}{z+2}\right| = \dfrac{\pi}{6}$, then the locus of z is ....

(x) If $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$, then $z =$ ....

26. State True or False for the following :

(i) The order relation is defined on the set of complex numbers.

(ii) Multiplication of a non zero complex number by $-i$ rotates the point about origin through a right angle in the anti-clockwise direction.

(iii) For any complex number z the minimum value of $|z| + |z-1|$ is 1.

(iv) The locus represented by $|z-1| = |z-i|$ is a line perpendicular to the join of (1, 0) and (0, 1).

(v) If z is a complex number such that $z \neq 0$ and Re $(z) = 0$, then Im $(z^2) = 0$.

(vi) The inequality $|z-4| < |z-2|$ represents the region given by $x > 3$.

(vii) Let z_1 and z_2 be two complex numbers such that $|z_1+z_2|=|z_1|+|z_2|$, then $\arg(z_1 - z_2) = 0$.

(viii) 2 is not a complex number.

27. Match the statements of Column A and Column B.

	Column A		Column B
(a)	The polar form of $i+\sqrt{3}$ is	(i)	Perpendicular bisector of segment joining (– 2, 0) and (2, 0)
(b)	The amplitude of $-1+\sqrt{-3}$ is	(ii)	On or outside the circle having centre at (0, – 4) and radius 3.
(c)	If $\lvert z+2\rvert=\lvert z-2\rvert$, then locus of z is	(iii)	$\frac{2\pi}{3}$
(d)	If $\lvert z+2i\rvert=\lvert z-2i\rvert$, then locus of z is	(iv)	Perpendicular bisector of segment joining (0, – 2) and (0, 2).
(e)	Region represented by $\lvert z+4i\rvert\geq 3$ is	(v)	$2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$
(f)	Region represented by $\lvert z+4\rvert\leq 3$ is	(vi)	On or inside the circle having centre (– 4, 0) and radius 3 units.
(g)	Conjugate of $\frac{1+2i}{1-i}$ lies in	(vii)	First quadrant
(h)	Reciprocal of $1-i$ lies in	(viii)	Third quadrant

28. What is the conjugate of $\frac{2-i}{(1-2i)^2}$?

29. If $|z_1|=|z_2|$, is it necessary that $z_1 = z_2$?

30. If $\frac{(a^2+1)^2}{2a-i} = x + iy$, what is the value of $x^2 + y^2$?

31. Find z if $|z| = 4$ and arg $(z) = \dfrac{5\pi}{6}$.

32. Find $\left|(1+i)\dfrac{(2+i)}{(3+i)}\right|$

33. Find principal argument of $(1 + i\sqrt{3})^2$.

34. Where does z lie, if $\left|\dfrac{z-5i}{z+5i}\right| = 1$.

Choose the correct answer from the given four options indicated against each of the Exercises from 35 to 50 (M.C.Q)

35. $\sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other for:

(A) $x = n\pi$ (B) $x = \left(n+\dfrac{1}{2}\right)\dfrac{\pi}{2}$

(C) $x = 0$ (D) No value of x

36. The real value of α for which the expression $\dfrac{1-i\sin\alpha}{1+2i\sin\alpha}$ is purely real is :

(A) $(n+1)\dfrac{\pi}{2}$ (B) $(2n + 1)\dfrac{\pi}{2}$

(C) $n\pi$ (D) None of these, where $n \in \mathbf{N}$

37. If $z = x + iy$ lies in the third quadrant, then $\dfrac{\bar{z}}{z}$ also lies in the third quadrant if

(A) $x > y > 0$ (B) $x < y < 0$

(C) $y < x < 0$ (D) $y > x > 0$

38. The value of $(z + 3)(\bar{z} + 3)$ is equivalent to

(A) $|z+3|^2$ (B) $|z-3|$

(C) $z^2 + 3$ (D) None of these

39. If $\left(\dfrac{1+i}{1-i}\right)^x = 1$, then

(A) $x = 2n+1$ (B) $x = 4n$

(C) $x = 2n$ (D) $x = 4n + 1$, where $n \in \mathrm{N}$

40. A real value of x satisfies the equation $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta\ (\alpha, \beta \in \mathbf{R})$ if $\alpha^2 + \beta^2 =$

(A) 1 (B) – 1 (C) 2 (D) – 2

41. Which of the following is correct for any two complex numbers z_1 and z_2?

(A) $|z_1 z_2| = |z_1||z_2|$ (B) $\arg(z_1 z_2) = \arg(z_1).\arg(z_2)$

(C) $|z_1 + z_2| = |z_1| + |z_2|$ (D) $|z_1 + z_2| \geq |z_1| - |z_2|$

42. The point represented by the complex number $2 - i$ is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is:

(A) $1 + 2i$ (B) $-1 - 2i$ (C) $2 + i$ (D) $-1 + 2i$

43. Let $x, y \in \mathbf{R}$, then $x + iy$ is a non real complex number if:

(A) $x = 0$ (B) $y = 0$ (C) $x \neq 0$ (D) $y \neq 0$

44. If $a + ib = c + id$, then

(A) $a^2 + c^2 = 0$ (B) $b^2 + c^2 = 0$

(C) $b^2 + d^2 = 0$ (D) $a^2 + b^2 = c^2 + d^2$

45. The complex number z which satisfies the condition $\left|\frac{i+z}{i-z}\right| = 1$ lies on

(A) circle $x^2 + y^2 = 1$ (B) the x-axis

(C) the y-axis (D) the line $x + y = 1$.

46. If z is a complex number, then

(A) $|z^2| > |z|^2$ (B) $|z^2| = |z|^2$

(C) $|z^2| < |z|^2$ (D) $|z^2| \geq |z|^2$

47. $|z_1 + z_2| = |z_1| + |z_2|$ is possible if

(A) $z_2 = \overline{z_1}$ (B) $z_2 = \frac{1}{z_1}$

(C) $\arg(z_1) = \arg(z_2)$ (D) $|z_1| = |z_2|$

48. The real value of θ for which the expression $\frac{1+i\cos\theta}{1-2i\cos\theta}$ is a real number is:

(A) $n\pi+\frac{\pi}{4}$ (B) $n\pi+(-1)^n\frac{\pi}{4}$

(C) $2n\pi\pm\frac{\pi}{2}$ (D) none of these.

49. The value of arg (x) when $x < 0$ is:

(A) 0 (B) $\frac{\pi}{2}$

(C) π (D) none of these

50. If $f(z) = \frac{7-z}{1-z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is

(A) $\frac{|z|}{2}$ (B) $|z|$

(C) $2|z|$ (D) none of these.

Chapter 6

LINEAR INEQUALITIES

6.1 Overview

6.1.1 A statement involving the symbols '>', '<', '≥', '≤' is called an inequality. For example $5 > 3$, $x \leq 4$, $x + y \geq 9$.

(i) Inequalities which do not involve variables are called numerical inequalities. For example $3 < 8$, $5 \geq 2$.

(ii) Inequalities which involve variables are called literal inequalities. For example, $x > 3$, $y \leq 5$, $x - y \geq 0$.

(iii) An inequality may contain more than one variable and it can be linear, quadratic or cubic etc. For eaxmple, $3x - 2 < 0$ is a linear inequality in one variable, $2x + 3y \geq 4$ is a linear inequality in two variables and $x^2 + 3x + 2 < 0$ is a quadratic inequality in one variable.

(iv) Inequalities involving the symbol '>' or '<' are called **strict inequalities**. For example, $3x - y > 5$, $x < 3$.

(v) Inequalities involving the symbol '≥' or '≤' are called **slack inequalities**. For example, $3x - y \geq 5$, $x \leq 5$.

6.1.2 *Solution of an inequality*

(i) The value(s) of the variable(s) which makes the inequality a true statement is called its **solutions**. The set of all solutions of an inequality is called the **solution set** of the inequality. For example, $x - 1 \geq 0$, has infinite number of solutions as all real values greater than or equal to one make it a true statement. The inequality $x^2 + 1 < 0$ has no solution in **R** as no real value of x makes it a true statement.

To solve an inequality we can

(i) Add (or subtract) the same quantity to (from) both sides without changing the sign of inequality.

(ii) Multiply (or divide) both sides by the same positive quantity without changing the sign of inequality. However, if both sides of inequality are multiplied (or divided) by the same negative quantity the sign of inequality is reversed, i.e., '>' changes into '<' and vice versa.

6.1.3 *Representation of solution of linear inequality in one variable on a number line*

To represent the solution of a linear inequality in one variable on a number line, we use the following conventions:

(i) If the inequality involves '≥' or '≤', we draw filled circle (•) on the number line to indicate that the number corresponding to the filled circle is included in the solution set.

(ii) If the inequality involves '>' or '<', we draw an open circle (O) on the number line to indicate that the number corresponding to the open circle is excluded from the solution set.

6.1.4 *Graphical representation of the solution of a linear inequality*

(a) To represent the solution of a linear inequality in one or two variables graphically in a plane, we proceed as follows:

(i) If the inequality involves '≥' or '≤', we draw the graph of the line as a thick line to indicate that the points on this line are included in the solution set.

(ii) If the inequality involves '>' or '<', we draw the graph of the line as dotted line to indicate that the points on the line are excluded from the solution set.

(b) Solution of a linear inequality in one variable can be represented on number line as well as in the plane but the solution of a linear inequality in two variables of the type $ax + by > c$, $ax + by \geq c$, $ax + by < c$ or $ax + by \leq c$ $(a \neq 0,\ b \neq 0)$ can be represented in the plane only.

(c) Two or more inequalities taken together comprise a system of inequalities and the solutions of the system of inequalities are the solutions common to all the inequalities comprising the system.

6.1.5 *Two important results*

(a) If $a, b \in \mathbf{R}$ and $b \neq 0$, then

(i) $ab > 0$ or $\dfrac{a}{b} > 0 \Rightarrow a$ and b are of the same sign.

(ii) $ab < 0$ or $\dfrac{a}{b} < 0 \Rightarrow a$ and b are of opposite sign.

(b) If a is any positive real number, i.e., $a > 0$, then

(i) $|x| < a \Leftrightarrow -a < x < a$

$|x| \leq a \Leftrightarrow -a \leq x \leq a$

(ii) $|x| > a \Leftrightarrow x < -a$ or $x > a$

$|x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$

6.2 Solved Examples

Short Answer Type

Example 1 Solve the inequality, $3x - 5 < x + 7$, when

(i) x is a natural number (ii) x is a whole number

(iii) x is an integer (iv) x is a real number

Solution We have $3x - 5 < x + 7$

$\Rightarrow \quad 3x < x + 12$ (Adding 5 to both sides)

$\Rightarrow \quad 2x < 12$ (Subtracting x from both sides)

$\Rightarrow \quad x < 6$ (Dividing by 2 on both sides)

(i) Solution set is $\{1, 2, 3, 4, 5\}$

(ii) Solution set is $\{0, 1, 2, 3, 4, 5\}$

(iii) Solution set is $\{....-3, -2, -1, 0, 1, 2, 3, 4, 5\}$

(iv) Solution set is $\{x : x \in \mathbf{R}$ and $x < 6\}$, i.e., any real number less than 6.

Example 2 Solve $\dfrac{x-2}{x+5} > 2$

Solution We have $\dfrac{x-2}{x+5} > 2$

$\Rightarrow \quad \dfrac{x-2}{x+5} - 2 > 0$ [Subtracting 2 from each side]

$\Rightarrow \quad \dfrac{-(x+12)}{x+5} > 0$

$\Rightarrow \quad \dfrac{x+12}{x+5} < 0$ (Multiplying both sides by – 1)

$\Rightarrow \quad x + 12 > 0$ and $x + 5 < 0$ [Since $\dfrac{a}{b} < 0 \Rightarrow a$ and b are of opposite signs]

or

$x + 12 < 0$ and $x + 5 > 0$

$\Rightarrow \quad x > -12$ and $x < -5$

or

$x < -12$ and $x > -5$ (Not possible)

Therefore, $-12 < x < -5$, i.e. $x \in (-12, -5)$

Example 3 Solve $|3-4x| \geq 9$.

Solution We have $|3-4x| \geq 9$.

$\Rightarrow \quad 3 - 4x \leq -9$ or $3 - 4x \geq 9$ (Since $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a$)

$\Rightarrow \quad -4x \leq -12$ or $-4x \geq 6$

$\Rightarrow \quad x \geq 3 \quad$ or $\quad x \leq \dfrac{-3}{2}$ (Dividing both sides by -4)

$\Rightarrow \quad x \in (-\infty, \dfrac{-3}{2}] \cup [3, \infty)$

Example 4 Solve $1 \leq |x-2| \leq 3$.

Solution We have $1 \leq |x - 2| \leq 3$

$\Rightarrow \quad |x-2| \geq 1$ and $|x-2| \leq 3$

$\Rightarrow \quad (x - 2 \leq -1$ or $x - 2 \geq 1)$ and $(-3 \leq x - 2 \leq 3)$

$\Rightarrow \quad (x \leq 1$ or $x \geq 3)$ and $(-1 \leq x \leq 5)$

$\Rightarrow \quad x \in (-\infty, 1] \cup [3, \infty)$ and $x \in [-1, 5]$

Combining the solutions of two inequalities, we have

$x \in [-1, 1] \cup [3, 5]$

Example 5 The cost and revenue functions of a product are given by $C(x) = 20x + 4000$ and $R(x) = 60x + 2000$, respectively, where x is the number of items produced and sold. How many items must be sold to realise some profit?

Solution We have, profit = Revenue – Cost

$= (60x + 2000) - (20x + 4000)$

$= 40x - 2000$

To earn some profit, $40x - 2000 > 0$

$\Rightarrow \quad x > 50$

Hence, the manufacturer must sell more than 50 items to realise some profit.

Example 6 Solve for x, $|x+1| + |x| > 3$.

Solution On LHS of the given inequality, we have two terms both containing modulus. By equating the expression within the modulus to zero, we get $x = -1, 0$ as critical points. These critical points divide the real line in three parts as $(-\infty, -1)$, $[-1, 0)$, $[0, \infty)$.

Case I When $-\infty < x < -1$

$$|x+1| + |x| > 3 \Rightarrow -x-1-x > 3 \Rightarrow x < -2.$$

Case II When $-1 \leq x < 0$,

$$|x+1| + |x| > 3 \Rightarrow x+1-x > 3 \Rightarrow 1 > 3 \qquad \text{(not possible)}$$

Case III When $0 \leq x < \infty$,

$$|x+1| + |x| > 3 \Rightarrow x+1+x > 3 \Rightarrow x > 1.$$

Combining the results of cases (I), (II) and (III), we get

$$x \in (-\infty, -2) \cup (1, \infty)$$

Long Answer Type

Example 7 Solve for x, $\dfrac{|x+3| + x}{x+2} > 1$

Solution We have $\dfrac{|x+3| + x}{x+2} > 1$

$$\Rightarrow \quad \frac{|x+3| + x}{x+2} - 1 > 0$$

$$\Rightarrow \quad \frac{|x+3| - 2}{x+2} > 0$$

Now two cases arise:

Case I When $x + 3 \geq 0$, i.e., $x \geq -3$. Then

$$\frac{|x+3| - 2}{x+2} > 0 \Rightarrow \frac{x+3-2}{x+2} > 0$$

$$\Rightarrow \quad \frac{x+1}{x+2} > 0$$

$$\Rightarrow \quad \{(x+1) > 0 \text{ and } x+2 > 0\} \quad \text{or} \quad \{x+1 < 0 \text{ and } x+2 < 0\}$$

$$\Rightarrow \quad \{x > -1 \text{ and } x > -2\} \quad \text{or} \quad \{x < -1 \text{ and } x < -2\}$$

$$\Rightarrow \quad x > -1 \quad \text{or} \quad x < -2$$

$$\Rightarrow \quad x \in (-1, \infty) \quad \text{or} \quad x \in (-\infty, -2)$$

$$\Rightarrow \quad x \in (-3, -2) \cup (-1, \infty) \quad [\text{Since } x \geq -3] \qquad \ldots (1)$$

Case II When $x + 3 < 0$, i.e., $x < -3$

$$\frac{|x+3|-2}{x+2} > 0 \quad \Rightarrow \quad \frac{-x-3-2}{x+2} > 0$$

$$\Rightarrow \quad \frac{-(x+5)}{x+2} > 0 \quad \Rightarrow \quad \frac{x+5}{x+2} < 0$$

$\Rightarrow$ $(x + 5 < 0 \text{ and } x + 2 > 0)$ or $(x + 5 > 0 \text{ and } x + 2 < 0)$

$\Rightarrow$ $(x < -5 \text{ and } x > -2)$ or $(x > -5 \text{ and } x < -2)$

it is not possible.

$\Rightarrow$ $x \in (-5, -2)$ ... (2)

Combining (I) and (II), the required solution is

$$x \in (-5, -2) \cup (-1, \infty)$$

Example 8 Solve the following system of inequalities :

$$\frac{x}{2x+1} \geq \frac{1}{4}, \quad \frac{6x}{4x-1} < \frac{1}{2}$$

Solution From the first inequality, we have $\dfrac{x}{2x+1} - \dfrac{1}{4} \geq 0$

$\Rightarrow$ $\dfrac{2x-1}{2x+1} \geq 0$

$\Rightarrow$ $(2x - 1 \geq 0 \text{ and } 2x + 1 > 0)$ or $(2x - 1 \leq 0 \text{ and } 2x + 1 < 0)$ [Since $2x + 1 \neq 0$)

$\Rightarrow$ $(x \geq \dfrac{1}{2} \text{ and } x > -\dfrac{1}{2})$ or $(x \leq \dfrac{1}{2} \text{ and } x < -\dfrac{1}{2})$

$\Rightarrow$ $x \geq \dfrac{1}{2}$ or $x < -\dfrac{1}{2}$

$\Rightarrow$ $x \in (-\infty, -\dfrac{1}{2}) \cup [\dfrac{1}{2}, \infty)$ ... (1)

From the second inequality, we have $\dfrac{6x}{4x-1} - \dfrac{1}{2} < 0$

$\Rightarrow$ $\dfrac{8x+1}{4x-1} < 0$

$\Rightarrow$ $(8x + 1 < 0 \text{ and } 4x - 1 > 0)$ or $(8x + 1 > 0 \text{ and } 4x - 1 < 0)$

$\Rightarrow$ $(x < -\dfrac{1}{8} \text{ and } x > \dfrac{1}{4})$ or $(x > -\dfrac{1}{8} \text{ and } x < \dfrac{1}{4})$

$\Rightarrow$ $x \in (-\dfrac{1}{8}, \dfrac{1}{4})$ (Since the first is not possible) ... (2)

Note that the common solution of (1) and (2) is null set. Hence, the given system of inequalities has no solution.

Example 9 Find the linear inequalities for which the shaded region in the given figure is the solution set.

Solution

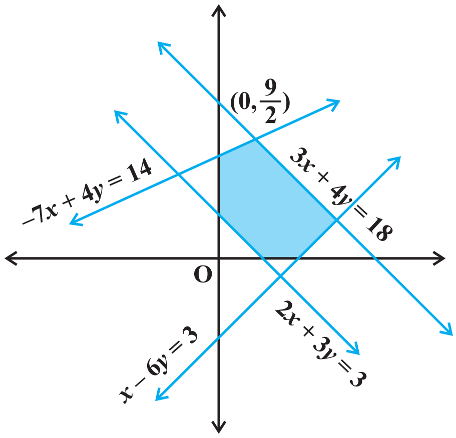


Fig 6.1

(i) Consider $2x + 3y = 3$. We observe that the shaded region and the origin lie on opposite side of this line and (0, 0) satisfies $2x + 3y \leq 3$. Therefore, we must have $2x + 3y \geq 3$ as linear inequality corresponding to the line $2x + 3y = 3$.

(ii) Consider $3x + 4y = 18$. We observe that the shaded region and the origin lie on the same side of this line and (0, 0) satisfies $3x + 4y \leq 18$. Therefore, $3x + 4y \leq 18$ is the linear inequality corresponding to the line $3x + 4y = 18$.

(iii) Consider $-7x + 4y = 14$. It is clear from the figure that the shaded region and the origin lie on the same side of this line and (0, 0) satisfies the inequality $-7x + 4y \leq 14$. Therefore, $-7x + 4y \leq 14$ is the inequality corresponding to the line $-7x + 4y = 14$.

(iv) Consider $x - 6y = 3$. It may be noted that the shaded portion and origin lie on the same side of this line and (0, 0) satisfies $x - 6y \leq 3$. Therefore, $x - 6y \leq 3$ is the inequality corresponding to the line $x - 6y = 3$.

(v) Also the shaded region lies in the first quadrant only. Therefore, $x \geq 0$, $y \geq 0$.

Hence, in view of (i), (ii), (iii), (iv) and (v) above, the linear inequalities corresponding to the given solution set are :

$2x + 3y \geq 3, 3x + 4y \leq 18\ -7x + 4y \leq 14, x - 6y \leq 3, x \geq 0, y \geq 0.$

Objective Type

Choose the correct answer from the given four options against each of the Examples 10 to 13 (M.C.Q.)

Example 10 If $\frac{|x-2|}{x-2} \geq 0$, then

(A) $x \in [2, \infty)$ (B) $x \in (2, \infty)$ (C) $x \in (-\infty, 2)$ (D) $x \in (-\infty, 2]$

Solution (B) is the correct choice. Since $\frac{|x-2|}{x-2} \geq 0$, for $|x-2| \geq 0$, and $x - 2 \neq 0$.

Example 11 The length of a rectangle is three times the breadth. If the minimum perimeter of the rectangle is 160 cm, then

(A) breadth > 20 cm (B) length < 20 cm

(C) breadth $x \geq 20$ cm (D) length ≤ 20 cm

Solution (C) is the correct choice. If x cm is the breadth, then

$$2\,(3x + x) \geq 160 \Rightarrow x \geq 20$$

Example 12 Solutions of the inequalities comprising a system in variable x are represented on number lines as given below, then

Fig 6.2

(A) $x \in (-\infty, -4] \cup [3, \infty)$ (B) $x \in [-3, 1]$

(C) $x \in (-\infty, -4) \cup [3, \infty)$ (D) $x \in [-4, 3]$

Solution (A) is the correct choice

Common solution of the inequalities is from $-\infty$ to -4 and 3 to ∞.

Example 13 If $|x+3| \geq 10$, then

(A) $x \in (-13, 7]$ (B) $x \in (-13, 7]$

(C) $x \in (-\infty, -13] \cup [7, \infty)$ (D) $x \in [-\infty, -13] \cup [7, \infty)$

Solution (D) is the correct choice, since $|x+3| \geq 10$, $\Rightarrow x + 3 \leq -10$ or $x + 3 \geq 10$

$\Rightarrow$ $x \leq -13$ or $x \geq 7$

$\Rightarrow$ $x \in (-\infty, -13] \cup [7, \infty)$

Example 14 State whether the following statements are True or False.

(i) If $x > y$ and $b < 0$, then $bx < by$

(ii) If $xy > 0$, then $x > 0$, and $y < 0$

(iii) If $xy < 0$, then $x > 0$, and $y > 0$

(iv) If $x > 5$ and $x > 2$, then $x \in (5, \infty)$

(v) If $|x| < 5$, then $x \in (-5, 5)$

(vi) Graph of $x > -2$ is

(vii) Solution set of $x - y \leq 0$ is

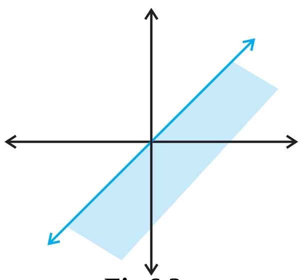


Fig 6.3

Solution

(i) True, because the sign of inequality is reversed when we multiply both sides of an inequality by a negative quantity.

(ii) False, product of two numbers is positive if they have the same sign.

(iii) False, product of two numbers is negative if they have opposite signs.

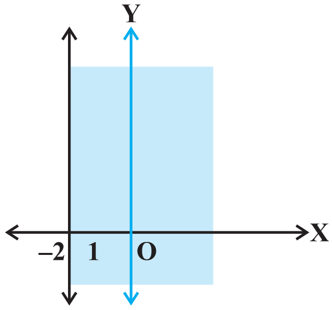


Fig 6.4

(iv) True (v)

True if $|x| < 5 \Rightarrow -5 < x < 5 \Rightarrow x \in (-5,$

$5)$.

(vi) False, because for $x > -2$, the line $x = -2$ has to be dotted, i.e., the region does not include the points on the line $x = -2$ (vii)

False, because (1, 0) does not satisfy the given inequality and it is a point in shaded portion.

Example 15 Fill in the blanks in the following:

(i) If $x \geq -3$, then $x + 5$ 2

(ii) If $-x \leq -4$, then $2x$ 8

(iii) If $\frac{1}{x-2} < 0$, then x 2

$\frac{a}{c}$ .... $\frac{b}{c}$

(v) If $|x-1| \leq 2$, then -1..... x 3

(vi) If $|3x - 7| > 2$, then x .... $\frac{5}{3}$ or x 3

(vii) If $p > 0$ and $q < 0$, then $p + q$ ... p

Solution

(i) ($\geq$), because same number can be added to both sides of inequality without changing the sign of inequality.

(ii) ($\geq$), after multiplying both sides by -2, the sign of inequality is reversed.

(iii) ($<$), because if $\frac{a}{b} < 0$ and $a > 0$, then $b < 0$.

(iv) ($>$), if both sides are divided by the same negative quantity, then the sign of

(v) $(\leq, \leq)$, $|x-1| \leq 2 \Rightarrow -2 \leq x - 1 \leq 2 \Rightarrow -1 \leq x \leq 3.$

(vi) $(<, >)$, $|3x - 7| > 2 \Rightarrow 3x - 7 < -2$ or $3x - 7 > 2$

$\Rightarrow x < \frac{5}{3}$ or $x > 3$

(vii) $(<)$, as p is positive and q is negative, therefore, $p + q$ is always smaller than p.

6.3 EXERCISE

Short Answer Type

Solve for x, the inequalities in Exercises 1 to 12.

1. $\frac{4}{x+1} \leq 3 \leq \frac{6}{x+1}, (x > 0)$

2. $\frac{|x-2|-1}{|x-2|-2} \leq 0$

3. $\frac{1}{|x|-3} \leq \frac{1}{2}$

4. $|x-1| \leq 5, \ |x| \geq 2$

5. $-5 \leq \frac{2-3x}{4} \leq 9$

6. $4x + 3 \geq 2x + 17, 3x - 5 < -2.$

7. A company manufactures cassettes. Its cost and revenue functions are $C(x) = 26{,}000 + 30x$ and $R(x) = 43x$, respectively, where x is the number of cassettes produced and sold in a week. How many cassettes must be sold by the company to realise some profit?

8. The water acidity in a pool is considerd normal when the average pH reading of three daily measurements is between 8.2 and 8.5. If the first two pH readings are 8.48 and 8.35, find the range of pH value for the third reading that will result in the acidity level being normal.

9. A solution of 9% acid is to be diluted by adding 3% acid solution to it. The resulting mixture is to be more than 5% but less than 7% acid. If there is 460 litres of the 9% solution, how many litres of 3% solution will have to be added?

10. A solution is to be kept between 40°C and 45°C. What is the range of temperature in degree fahrenheit, if the conversion formula is $F = \frac{9}{5}C + 32$?

11. The longest side of a triangle is twice the shortest side and the third side is 2cm longer than the shortest side. If the perimeter of the triangle is more than 166 cm then find the minimum length of the shortest side.

12. In drilling world's deepest hole it was found that the temperature T in degree celcius, x km below the earth's surface was given by $T = 30 + 25 (x - 3)$, $3 \leq x \leq 15$. At what depth will the temperature be between 155°C and 205°C?

Long Answer Type

13. Solve the following system of inequalities $\frac{2x+1}{7x-1} > 5$, $\frac{x+7}{x-8} > 2$

14. Find the linear inequalities for which the shaded region in the given figure is the solution set.

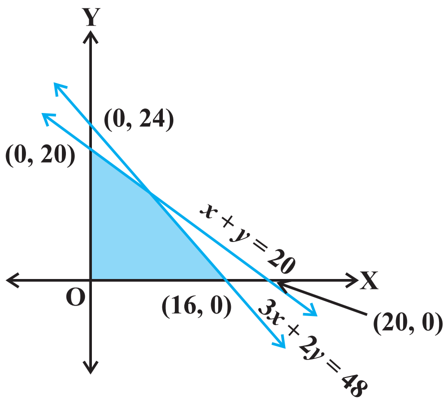


Fig 6.5

15. Find the linear inequalities for which the shaded region in the given figure is the solution set.

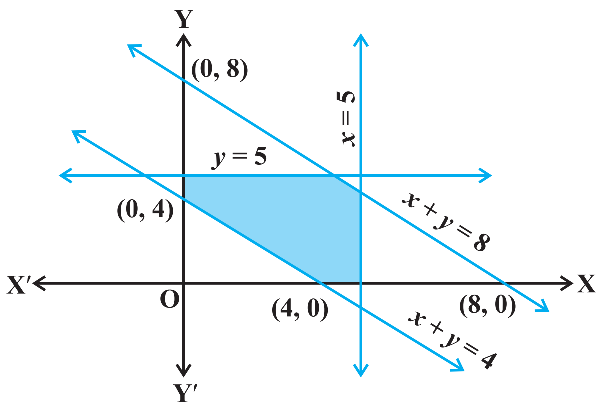


Fig 6.6

16. Show that the following system of linear inequalities has no solution

$x + 2y \leq 3,\ 3x + 4y \geq 12,\ x \geq 0,\ y \geq 1$

17. Solve the following system of linear inequalities:

$3x + 2y \geq 24$, $3x + y \leq 15$, $x \geq 4$

18. Show that the solution set of the following system of linear inequalities is an unbounded region

$2x + y \geq 8$, $x + 2y \geq 10$, $x \geq 0$, $y \geq 0$

Objective Type Question

Choose the correct answer from the given four options in each of the Exercises 19 to 26 (M.C.Q.).

19. If $x < 5$, then

(A) $-x < -5$ (B) $-x \leq -5$

(C) $-x > -5$ (D) $-x \geq -5$

20. Given that x, y and b are real numbers and $x < y$, $b < 0$, then

(A) $\frac{x}{b} < \frac{y}{b}$ (B) $\frac{x}{b} \leq \frac{y}{b}$

(C) $\frac{x}{b} > \frac{y}{b}$ (D) $\frac{x}{b} \geq \frac{y}{b}$

21. If $-3x + 17 < -13$, then

(A) $x \in (10, \infty)$ (B) $x \in [10, \infty)$

(C) $x \in (-\infty, 10]$ (D) $x \in [-10, 10)$

22. If x is a real number and $|x| < 3$, then

(A) $x \geq 3$ (B) $-3 < x < 3$

(C) $x \leq -3$ (D) $-3 \leq x \leq 3$

23. x and b are real numbers. If $b > 0$ and $|x| > b$, then

(A) $x \in (-b, \infty)$ (B) $x \in [-\infty, b)$

(C) $x \in (-b, b)$ (D) $x \in (-\infty, -b) \cup (b, \infty)$

24. If $|x - 1| > 5$, then

(A) $x \in (-4, 6)$ (B) $x \in [-4, 6]$

(C) $x \in [-\infty, -4) \cup (6, \infty)$ (D) $x \in [-\infty, -4) \cup [6, \infty)$

25. If $|x + 2| \leq 9$, then

(A) $x \in (-7, 11)$ (B) $x \in [-11, 7]$

(C) $x \in (-\infty, -7) \cup (11, \infty)$ (D) $x \in (-\infty, -7) \cup [11, \infty)$

26. The inequality representing the following graph is:

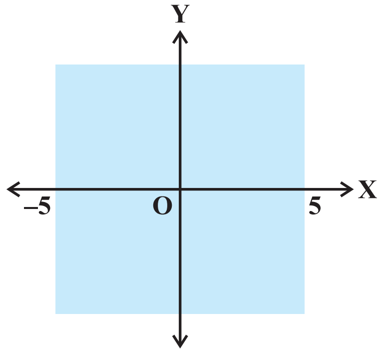


Fig 6.7

(A) $|x| < 5$ (B) $|x| \le 5$ (C) $|x| > 5$ (D) $|x| \ge 5$

Solution of a linear inequality in variable x is represented on number line in Exercises 27 to 30. Choose the correct answer from the given four options in each of the exercises (M.C.Q.).

27. (A) $x \in (-\infty, 5)$ (B) $x \in (-\infty, 5]$

(C) $x \in [5, \infty,)$ (D) $x \in (5, \infty)$

Fig 6.8

28. (A) $x \in (\frac{9}{2}, \infty)$

(B) $x \in [\frac{9}{2}, \infty)$

(D) $x \in [-\infty, \frac{9}{2})$

(D) $x \in (-\infty, \frac{9}{2}]$

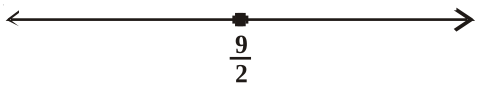


Fig 6.9

29. (A) $x \in (-\infty, \frac{7}{2})$ (B) $x \in (-\infty, \frac{7}{2}]$

(C) $x \in [\frac{7}{2}, -\infty)$ (D) $x \in (\frac{7}{2}, \infty)$

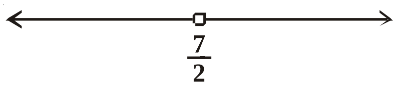


Fig 6.10

30. (A) $x \in (-\infty, -2)$

(B) $x \in (-\infty, -2]$

(C) $x \in (-2, \infty]$

(D) $x \in [-2, \infty)$

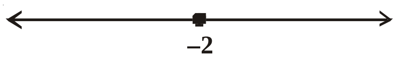


Fig 6.11

31. State which of the following statements is True or False

(i) If $x < y$ and $b < 0$, then $\frac{x}{b} < \frac{y}{b}$.

(ii) If $xy > 0$, then $x > 0$ and $y < 0$

(iii) If $xy > 0$, then $x < 0$ and $y < 0$

(iv) If $xy < 0$, then $x < 0$ and $y < 0$

(v) If $x < -5$ and $x < -2$, then $x \in (-\infty, -5)$

(vi) If $x < -5$ and $x > 2$, then $x \in (-5, 2)$

(vii) If $x > -2$ and $x < 9$, then $x \in (-2, 9)$

(viii) If $|x| > 5$, then $x \in (-\infty, -5) \cup [5, \infty)$

(ix) If $|x| \leq 4$, then $x \in [-4, 4]$

(x) Graph of $x < 3$ is

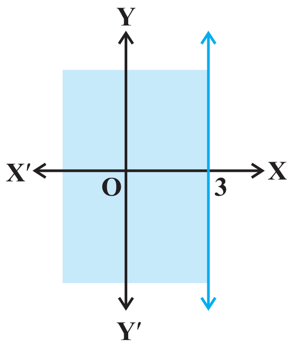


Fig 6.12

(xi) Graph of $x \geq 0$ is

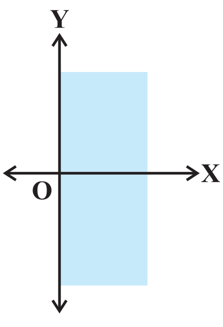


Fig 6.13

(xii) Graph of $y \leq 0$ is

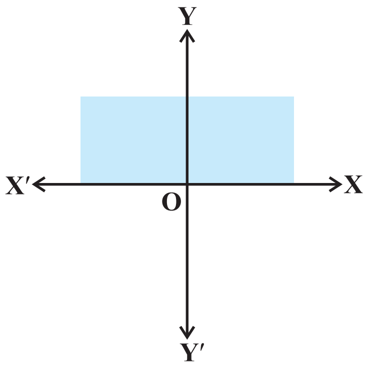


Fig 6.14

(xiii) Solution set of $x \geq 0$ and $y \leq 0$ is

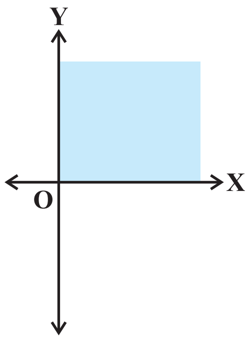


Fig 6.15

(xiv) Solution set of $x \geq 0$ and $y \leq 1$ is

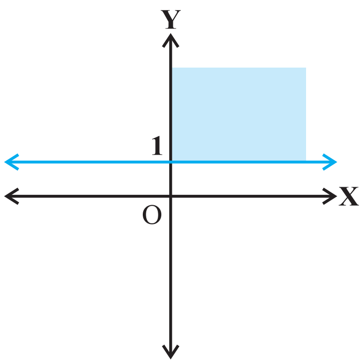


Fig 6.16

(xv) Solution set of $x + y \geq 0$ is

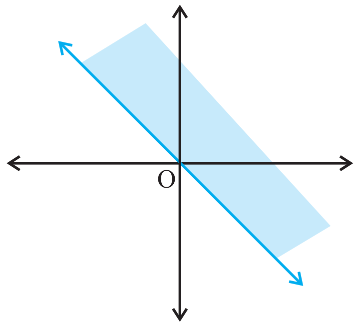


Fig 6.17

32. Fill in the blanks of the following:

(i) If $-4x \geq 12$, then x ... -3.

(ii) If $\frac{-3}{4}x \leq -3$, then x ... 4.

(iii) If $\frac{2}{x+2} > 0$, then x ... -2.

(iv) If $x > -5$, then $4x$ ... -20.

(v) If $x > y$ and $z < 0$, then $-xz$ ... $-yz$.

(vi) If $p > 0$ and $q < 0$, then $p - q$ ... p.

(vii) If $|x+2| > 5$, then x ... -7 or x ... 3.

(viii) If $-2x + 1 \geq 9$, then x ... -4.

Chapter 7

PERMUTATIONS AND COMBINATIONS

7.1 Overview

The study of permutations and combinations is concerned with determining the number of different ways of arranging and selecting objects out of a given number of objects, without actually listing them. There are some basic counting techniques which will be useful in determining the number of different ways of arranging or selecting objects. The two basic counting principles are given below:

Fundamental principle of counting

7.1.1 *Multiplication principle (Fundamental Principle of Counting)*

Suppose an event E can occur in m different ways and associated with each way of occurring of E, another event F can occur in n different ways, then the total number of occurrence of the two events in the given order is $m \times n$.

7.1.2 *Addition principle*

If an event E can occur in m ways and another event F can occur in n ways, and suppose that both can not occur together, then E or F can occur in $m + n$ ways.

7.1.3 ***Permutations*** A permutation is an arrangement of objects in a definite order.

7.1.4 ***Permutation of n different objects*** The number of permutations of n objects taken all at a time, denoted by the symbol ${}^nP_n$, is given by

$$ {}^nP_n = \lfloor n , \qquad \ldots (1)$$

where $\lfloor n = n(n-1)(n-2) \ldots 3.2.1$, read as factorial n, or n factorial.

The number of permutations of n objects taken r at a time, where $0 < r \leq n$, denoted by ${}^nP_r$, is given by

$$ {}^nP_r = \frac{\lfloor n}{\lfloor n-r} $$

We assume that $\lfloor 0 = 1$

7.1.5 ***When repetition of objects is allowed*** The number of permutations of n things taken all at a time, when repetion of objects is allowed is n^n.

The number of permutations of n objects, taken r at a time, when repetition of objects is allowed, is n^r.

7.1.6 ***Permutations when the objects are not distinct*** The number of permutations of n objects of which p_1 are of one kind, p_2 are of second kind, ..., p_k are of k^{th} kind and the rest if any, are of different kinds is $\dfrac{n!}{p_1!\,p_2!\ldots p_k!}$

7.1.7 ***Combinations*** On many occasions we are not interested in arranging but only in selecting r objects from given n objects. A combination is a selection of some or all of a number of different objects where the order of selection is immaterial. The number of selections of r objects from the given n objects is denoted by ${}^nC_r$, and is given by

$$ {}^nC_r = \frac{n!}{r!(n-r)!} $$

Remarks

1. Use permutations if a problem calls for the number of arrangements of objects and different orders are to be counted.
2. Use combinations if a problem calls for the number of ways of selecting objects and the order of selection is not to be counted.

7.1.8 ***Some important results***

Let n and r be positive integers such that $r \le n$. Then

(i) ${}^nC_r = {}^nC_{n-r}$

(ii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) $n\ {}^{n-1}C_{r-1} = (n-r+1)\ {}^nC_{r-1}$

7.2 Solved Examples

Short Answer Type

Example 1 In a class, there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent the class for a function. In how many ways can the teacher make this selection?

Solution Here the teacher is to perform two operations:

(i) Selecting a boy from among the 27 boys and

(ii) Selecting a girl from among 14 girls.

The first of these can be done in 27 ways and second can be performed in 14 ways. By the fundamental principle of counting, the required number of ways is $27 \times 14 = 378$.

Example 2

(i) How many numbers are there between 99 and 1000 having 7 in the units place?

(ii) How many numbers are there between 99 and 1000 having atleast one of their digits 7?

Solution

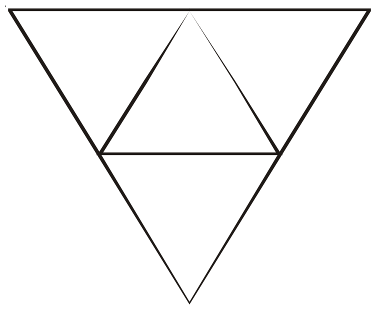
(i) First note that all these numbers have three digits. 7 is in the unit's place. The middle digit can be any one of the 10 digits from 0 to 9. The digit in hundred's place can be any one of the 9 digits from 1 to 9. Therefore, by the fundamental principle of counting, there are $10 \times 9 = 90$ numbers between 99 and 1000 having 7 in the unit's place.

(ii) Total number of 3 digit numbers having atleast one of their digits as 7 = (Total numbers of three digit numbers) – (Total number of 3 digit numbers in which 7 does not appear at all).
$= (9 \times 10 \times 10) - (8 \times 9 \times 9)$
$= 900 - 648 = 252$.

Example 3 In how many ways can this diagram be coloured subject to the following two conditions?

(i) Each of the smaller triangle is to be painted with one of three colours: red, blue or green.

(ii) No two adjacent regions have the same colour.



Solution These conditions are satisfied exactly when we do as follows: First paint the central triangle in any one of the three colours. Next paint the remaining 3 triangles, with any one of the remaining two colours.
By the fundamental principle of counting, this can be done in $3 \times 2 \times 2 \times 2 = 24$ ways.

Example 4 In how many ways can 5 children be arranged in a line such that (i) two particular children of them are always together (ii) two particular children of them are never together.

Solution

(i) We consider the arrangements by taking 2 particular children together as one and hence the remaining 4 can be arranged in $4! = 24$ ways. Again two particular children taken together can be arranged in two ways. Therefore, there are $24 \times 2 = 48$ total ways of arrangement.

(ii) Among the $5! = 120$ permutations of 5 children, there are 48 in which two children are together. In the remaining $120 - 48 = 72$ permutations, two particular children are never together.

Example 5 If all permutations of the letters of the word AGAIN are arranged in the order as in a dictionary. What is the 49^{th} word?

Solution Starting with letter A, and arranging the other four letters, there are $4! = 24$ words. These are the first 24 words. Then starting with G, and arranging A, A, I and N in different ways, there are $\frac{4!}{2!1!1!} = 12$ words. Next the 37^{th} word starts with I. There are again 12 words starting with I. This accounts up to the 48^{th} word. The 49^{th} word is NAAGI.

Example 6 In how many ways 3 mathematics books, 4 history books, 3 chemistry books and 2 biology books can be arranged on a shelf so that all books of the same subjects are together.

Solution First we take books of a particular subject as one unit. Thus there are 4 units which can be arranged in $4! = 24$ ways. Now in each of arrangements, mathematics books can be arranged in $3!$ ways, history books in $4!$ ways, chemistry books in $3!$ ways and biology books in $2!$ ways. Thus the total number of ways $= 4! \times 3! \times 4! \times 3! \times 2! = 41472$.

Example 7 A student has to answer 10 questions, choosing atleast 4 from each of Parts A and B. If there are 6 questions in Part A and 7 in Part B, in how many ways can the student choose 10 questions?

Solution The possibilities are:

4 from Part A and 6 from Part B

or 5 from Part A and 5 from Part B

or 6 from Part A and 4 from Part B.

Therefore, the required number of ways is

$$^6C_4 \times {}^7C_6 + {}^6C_5 \times {}^7C_5 + {}^6C_6 \times {}^7C_4$$
$$= 105 + 126 + 35 = 266.$$

Long Answer Type

Example 8 Suppose m men and n women are to be seated in a row so that no two women sit together. If $m > n$, show that the number of ways in which they can be seated is

$$\frac{m!(m+1)!}{(m-n+1)!}$$

Solution Let the men take their seats first. They can be seated in ^mP_m ways as shown in the following figure

□ M □ M □ ... □ M □

1^{st} 2^{nd} m^{th}

From the above figure, we observe, that there are $(m + 1)$ places for n women. It is given that $m > n$ and no two women can sit together. Therefore, n women can take their seats $^{(m+1)}P_n$ ways and hence the total number of ways so that no two women sit together is

$$({}^mP_m) \times ({}^{m+1}P_n) = \frac{m!(m+1)!}{(m-n+1)!}$$

Example 9 Three married couples are to be seated in a row having six seats in a cinema hall. If spouses are to be seated next to each other, in how many ways can they be seated? Find also the number of ways of their seating if all the ladies sit together.

Solution Let us denote married couples by S_1, S_2, S_3, where each couple is considered to be a single unit as shown in the following figure:

□□ □□ □□

1^{st} 2^{nd} 3^{rd}

Then the number of ways in which spouces can be seated next to each other is $3! = 6$ ways.

Again each couple can be seated in $2!$ ways. Thus the total number of seating arrangement so that spouces sit next to each other $= 3! \times 2! \times 2! \times 2! = 48$.

Again, if three ladies sit together, then necessarily three men must sit together. Thus, ladies and men can be arranged altogether among themselves in $2!$ ways. Therefore, the total number of ways where ladies sit together is $3! \times 3! \times 2! = 144$.

Example 10 In a small village, there are 87 families, of which 52 families have atmost 2 children. In a rural development programme 20 families are to be chosen for assistance, of which atleast 18 families must have at most 2 children. In how many ways can the choice be made?

Solution It is given that out of 87 families, 52 families have at most 2 children so other 35 families are of other type. According to the question, for rural development programme, 20 families are to be chosen for assistance, of which at least 18 families must have atmost 2 children. Thus, the following are the number of possible choices:

${}^{52}C_{18} \times {}^{35}C_2$ (18 families having atmost 2 children and 2 selected from other type of families)

${}^{52}C_{19} \times {}^{35}C_1$ (19 families having at most 2 children and 1 selected from other type of families)

${}^{52}C_{20}$ (All selected 20 families having atmost 2 children)

Hence, the total number of possible choices is

$$ {}^{52}C_{18} \times {}^{35}C_2 + {}^{52}C_{19} \times {}^{35}C_1 + {}^{52}C_{20} $$

Example 11 A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Mathematics Part II, unless Mathematics Part I is also borrowed. In how many ways can he choose the three books to be borrowed?

Solution Let us make the following cases:

Case (i) Boy borrows Mathematics Part II, then he borrows Mathematics Part I also. So the number of possible choices is ${}^{6}C_1 = 6$.

Case (ii) Boy does not borrow Mathematics Part II, then the number of possible choices is $7C_3 = 35$.

Hence, the total number of possible choices is 35 + 6 = 41.

Example 12 Find the number of permutations of n different things taken r at a time such that two specific things occur together.

Solution A bundle of 2 specific things can be put in r places in $(r – 1)$ ways (Why?) and 2 things in the bundle can be arranged themselves into $\lfloor 2$ ways. Now $(n – 2)$ things will be arranged in $(r – 2)$ places in ${}^{n-2}P_{r-2}$ ways.

Thus, using the fundamental principle of counting, the required numer of permutations will be $\lfloor 2 \cdot (r-1) \cdot {}^{n-2}P_{r-2}$.

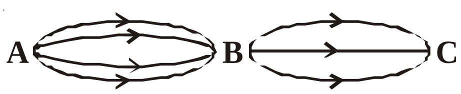
Objective Type Questions

Choose the correct answer out of four options given against each of the following Examples (M.C.Q.).

Example 13 There are four bus routes between A and B; and three bus routes between B and C. A man can travel round-trip in number of ways by bus from A to C via B. If he does not want to use a bus route more than once, in how many ways can he make round trip?

(A) 72 (B) 144 (C) 14 (D) 19

Solution (A) is the correct answer. In the following figure:



there are 4 bus routes from A to B and 3 routes from B to C. Therefore, there are $4 \times 3 = 12$ ways to go from A to C. It is round trip so the man will travel back from C to A via B. It is restricted that man can not use same bus routes from C to B and B to A more than once. Thus, there are $2 \times 3 = 6$ routes for return journey. Therefore, the required number of ways $= 12 \times 6 = 72$.

Example 14 In how many ways a committee consisting of 3 men and 2 women, can be chosen from 7 men and 5 women?

(A) 45 (B) 350 (C) 4200 (D) 230

Solution (B) is the correct choice. Out of 7 men, 3 men can be chosen in $7C_3$ ways and out of 5 women, 2 women can be chosen in $5C_2$ ways. Hence, the committee can be chosen in $7C_3 \times 5C_2 = 350$ ways.

Example 15 All the letters of the word 'EAMCOT' are arranged in different possible ways. The number of such arrangements in which no two vowels are adjacent to each other is

(A) 360 (B) 144 (C) 72 (D) 54

Solution (B) is the correct choice. We note that there are 3 consonants and 3 vowels E, A and O. Since no two vowels have to be together, the possible choice for vowels are the places marked as 'X'. X M X C X T X, these volwels can be arranged in ${}^4P_3$ ways 3 consonents can be arranged in $\lfloor\underline{3}$ ways. Hence, the required number of ways $= 3! \times {}^4P_3 = 144$.

Example 16 Ten different letters of alphabet are given. Words with five letters are formed from these given letters. Then the number of words which have atleast one letter repeated is

(A) 69760 (B) 30240 (C) 99748 (D) 99784

Solution (A) is correct choice. Number of 5 letters words (with the condition that a letter can be repeated) = 10^5. Again number of words using 5 different letters is ${}^{10}P_5$. Therefore, required number of letters

= Total number of words – Total number of words in which no letter is repeated

$= 10^5 - {}^{10}P_5 = 69760.$

Example 17 The number of signals that can be sent by 6 flags of different colours taking one or more at a time is

(A) 63 (B) 1956 (C) 720 (D) 21

Solution The correct answer is B.

Number of signals using one flag = ${}^6P_1 = 6$

Number of signals using two flags = ${}^6P_2 = 30$

Number of signals using three flags = ${}^6P_3 = 120$

Number of signals using four flags = ${}^6P_4 = 360$

Number of signals using five flags = ${}^6P_5 = 720$

Number of signals using all six flags = ${}^6P_6 = 720$

Therefore, the total number of signals using one or more flags at a time is

$$6 + 30 + 120 + 360 + 720 + 720 = 1956 \text{ (Using addition principle).}$$

Example 18 In an examination there are three multiple choice questions and each question has 4 choices. Number of ways in which a student can fail to get all answer correct is

(A) 11 (B) 12 (C) 27 (D) 63

Solution The correct choice is (D). There are three multiple choice question, each has four possible answers. Therefore, the total number of possible answers will be $4 \times 4 \times 4 = 64$. Out of these possible answer only one will be correct and hence the number of ways in which a student can fail to get correct answer is $64 - 1 = 63$.

Example 19 The straight lines l_1, l_2 and l_3 are parallel and lie in the same plane. A total numbers of m points are taken on l_1; n points on l_2, k points on l_3. The maximum number of triangles formed with vertices at these points are

(A) ${}^{(m+n+k)}C_3$ (B) ${}^{(m+n+k)}C_3 - {}^mC_3 - {}^nC_3 - {}^kC_3$

(C) ${}^mC_3 + {}^nC_3 + {}^kC_3$ (D) ${}^mC_3 \times {}^nC_3 \times {}^kC_3$

Solution (B) is the correct answer. Here the total number of points are $(m + n + k)$ which must give ${}^{(m+n+k)}C_3$ number of triangles but m points on l_1 taking 3 points at a time gives ${}^mC_3$ combinations which produce no triangle. Similarly, ${}^nC_3$ and ${}^kC_3$

number of triangles can not be formed. Therefore, the required number of triangles is $^{(m+n+k)}C_3 - {}^mC_3 - {}^nC_3 - {}^kC_3$.

7.3 EXERCISE

Short Answer Type

1. Eight chairs are numbered 1 to 8. Two women and 3 men wish to occupy one chair each. First the women choose the chairs from amongst the chairs 1 to 4 and then men select from the remaining chairs. Find the total number of possible arrangements.

   [**Hint:** 2 women occupy the chair, from 1 to 4 in $^4P_2$ ways and 3 men occupy the remaining chairs in $^6P_3$ ways.]

2. If the letters of the word RACHIT are arranged in all possible ways as listed in dictionary. Then what is the rank of the word RACHIT ?

   [**Hint:** In each case number of words beginning with A, C, H, I is 5!]

3. A candidate is required to answer 7 questions out of 12 questions, which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. Find the number of different ways of doing questions.

4. Out of 18 points in a plane, no three are in the same line except five points which are collinear. Find the number of lines that can be formed joining the point.

   [**Hint:** Number of straight lines = $^{18}C_2 - {}^5C_2 + 1$.]

5. We wish to select 6 persons from 8, but if the person A is chosen, then B must be chosen. In how many ways can selections be made?

6. How many committee of five persons with a chairperson can be selected from 12 persons.

   [**Hint:** Chairman can be selected in 12 ways and remaining in $^{11}C_4$.]

7. How many automobile license plates can be made if each plate contains two different letters followed by three different digits?

8. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected from the lot.

9. Find the number of permutations of n distinct things taken r together, in which 3 particular things must occur together.

10. Find the number of different words that can be formed from the letters of the word 'TRIANGLE' so that no vowels are together.

11. Find the number of positive integers greater than 6000 and less than 7000 which are divisible by 5, provided that no digit is to be repeated.

12. There are 10 persons named P_1, P_2, P_3, ... P_{10}. Out of 10 persons, 5 persons are to be arranged in a line such that in each arrangement P_1 must occur whereas P_4 and P_5 do not occur. Find the number of such possible arrangements.
[**Hint:** Required number of arrangement = ${}^7C_4 \times 5!$]

13. There are 10 lamps in a hall. Each one of them can be switched on independently. Find the number of ways in which the hall can be illuminated.
[**Hint:** Required number = $2^{10} - 1$].

14. A box contains two white, three black and four red balls. In how many ways can three balls be drawn from the box, if atleast one black ball is to be included in the draw.
[**Hint:** Required number of ways = ${}^3C_1 \times {}^6C_2 + {}^3C_2 \times {}^6C_2 + {}^3C_3$.]

15. If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then find ${}^rC_2$.
[**Hint:** Form equation using $\frac{{}^nC_r}{{}^nC_{r+1}}$ and $\frac{{}^nC_r}{{}^nC_{r-1}}$ to find the value of r.]

16. Find the number of integers greater than 7000 that can be formed with the digits 3, 5, 7, 8 and 9 where no digits are repeated.
[**Hint:** Besides 4 digit integers greater than 7000, five digit integers are always greater than 7000.]

17. If 20 lines are drawn in a plane such that no two of them are parallel and no three are concurrent, in how many points will they intersect each other?

18. In a certain city, all telephone numbers have six digits, the first two digits always being 41 or 42 or 46 or 62 or 64. How many telephone numbers have all six digits distinct?

19. In an examination, a student has to answer 4 questions out of 5 questions; questions 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.

20. A convex polygon has 44 diagonals. Find the number of its sides.
[**Hint:** Polygon of n sides has (${}^nC_2 - n$) number of diagonals.]

Long Answer Type Questions

21. 18 mice were placed in two experimental groups and one control group, with all groups equally large. In how many ways can the mice be placed into three groups?

22. A bag contains six white marbles and five red marbles. Find the number of ways in which four marbles can be drawn from the bag if (a) they can be of any colour (b) two must be white and two red and (c) they must all be of the same colour.

23. In how many ways can a football team of 11 players be selected from 16 players? How many of them will

(i) include 2 particular players?

(ii) exclude 2 particular players?

24. A sports team of 11 students is to be constituted, choosing at least 5 from Class XI and atleast 5 from Class XII. If there are 20 students in each of these classes, in how many ways can the team be constituted?

25. A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has

(i) no girls

(ii) at least one boy and one girl

(iii) at least three girls.

Objective Type Questions

Choose the correct answer out of the given four options against each of the Exercises from 26 to 40 (M.C.Q.).

26. If ${}^nC_{12} = {}^nC_8$, then n is equal to

(A) 20 (B) 12 (C) 6 (D) 30

27. The number of possible outcomes when a coin is tossed 6 times is

(A) 36 (B) 64 (C) 12 (D) 32

28. The number of different four digit numbers that can be formed with the digits 2, 3, 4, 7 and using each digit only once is

(A) 120 (B) 96 (C) 24 (D) 100

29. The sum of the digits in unit place of all the numbers formed with the help of 3, 4, 5 and 6 taken all at a time is

(A) 432 (B) 108 (C) 36 (D) 18

30. Total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is equal to

(A) 60 (B) 120 (C) 7200 (D) 720

31. A five digit number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 and 5 without repetitions. The total number of ways this can be done is

(A) 216 (B) 600 (C) 240 (D) 3125

[**Hint:** 5 digit numbers can be formed using digits 0, 1, 2, 4, 5 or by using digits 1, 2, 3, 4, 5 since sum of digits in these cases is divisible by 3.]

32. Every body in a room shakes hands with everybody else. The total number of hand shakes is 66. The total number of persons in the room is

(A) 11 (B) 12 (C) 13 (D) 14

33. The number of triangles that are formed by choosing the vertices from a set of 12 points, seven of which lie on the same line is

(A) 105 (B) 15 (C) 175 (D) 185

34. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is

(A) 6 (B) 18 (C) 12 (D) 9

35. The number of ways in which a team of eleven players can be selected from 22 players always including 2 of them and excluding 4 of them is

(A) ${}^{16}C_{11}$ (B) ${}^{16}C_5$ (C) ${}^{16}C_9$ (D) ${}^{20}C_9$

36. The number of 5-digit telephone numbers having atleast one of their digits repeated is

(A) 90,000 (B) 10,000 (C) 30,240 (D) 69,760

37. The number of ways in which we can choose a committee from four men and six women so that the committee includes at least two men and exactly twice as many women as men is

(A) 94 (B) 126 (C) 128 (D) None

38. The total number of 9 digit numbers which have all different digits is

(A) 10! (B) 9 ! (C) $9 \times 9!$ (D) $10\times10!$

39. The number of words which can be formed out of the letters of the word ARTICLE, so that vowels occupy the even place is

(A) 1440 (B) 144

(C) 7! (D) ${}^4C_4 \times {}^3C_3$

40. Given 5 different green dyes, four different blue dyes and three different red dyes, the number of combinations of dyes which can be chosen taking at least one green and one blue dye is

(A) 3600 (B) 3720 (C) 3800 (D) 3600

[**Hint:** Possible numbers of choosing or not choosing 5 green dyes, 4 blue dyes and 3 red dyes are 2^5, 2^4 and 2^3, respectively.]

Fill in the Blanks in the Exercises 41 to 50.

41. If ${}^nP_r = 840$, ${}^nC_r = 35$, then r = _______.

42. ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$ = _______.

43. The number of permutations of n different objects, taken r at a line, when repetitions are allowed, is _______.

44. The number of different words that can be formed from the letters of the word INTERMEDIATE such that two vowels never come together is _______.

[**Hint:** Number of ways of arranging 6 consonants of which two are alike is $\frac{6!}{2!}$ and number of ways of arranging vowels $= {}^7P_6 \times \frac{1}{3!} \times \frac{1}{2!}$.]

45. Three balls are drawn from a bag containing 5 red, 4 white and 3 black balls. The number of ways in which this can be done if at least 2 are red is _______

46. The number of six-digit numbers, all digits of which are odd is _______.

47. In a football championship, 153 matches were played . Every two teams played one match with each other. The number of teams, participating in the championship is _______.

48. The total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two signs '–' occur together is _______.

49. A committee of 6 is to be chosen from 10 men and 7 women so as to contain atleast 3 men and 2 women. In how many different ways can this be done if two particular women refuse to serve on the same committee.

[**Hint:** At least 3 men and 2 women: The number of ways = ${}^{10}C_3 \times {}^7C_3 + {}^{10}C_4 \times {}^7C_2$.
For 2 particular women to be always there: the number of ways = ${}^{10}C_4 + {}^{10}C_3 \times {}^5C_1$.
The total number of committees when two particular women are never together = Total – together.]

50. A box contains 2 white balls, 3 black balls and 4 red balls. The number of ways three balls be drawn from the box if at least one black ball is to be included in the draw is _______.

State whether the statements in Exercises from 51 to 59 True or False? Also give justification.

51. There are 12 points in a plane of which 5 points are collinear, then the number of lines obtained by joining these points in pairs is ${}^{12}C_2 - {}^5C_2$.

52. Three letters can be posted in five letterboxes in 3^5 ways.

53. In the permutations of n things, r taken together, the number of permutations in which m particular things occur together is ${}^{n-m}P_{r-m} \times {}^rP_m$.

54. In a steamer there are stalls for 12 animals, and there are horses, cows and calves (not less than 12 each) ready to be shipped. They can be loaded in 3^{12} ways.

55. If some or all of n objects are taken at a time, the number of combinations is $2^n - 1$.

56. There will be only 24 selections containing at least one red ball out of a bag containing 4 red and 5 black balls. It is being given that the balls of the same colour are identical.

57. Eighteen guests are to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on other side of the table. The number of ways in which the seating arrangements can be made is $\frac{11!}{5!6!}(9!)(9!)$.

[**Hint:** After sending 4 on one side and 3 on the other side, we have to select out of 11; 5 on one side and 6 on the other. Now there are 9 on each side of the long table and each can be arranged in 9! ways.]

58. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. He can choose the seven questions in 650 ways.

59. To fill 12 vacancies there are 25 candidates of which 5 are from scheduled castes. If 3 of the vacancies are reserved for scheduled caste candidates while the rest are open to all, the number of ways in which the selection can be made is ${}^5C_3 \times {}^{20}C_9$.

In each if the Exercises from 60 to 64 match each item given under the column C_1 to its correct answer given under the column C_2.

60. There are 3 books on Mathematics, 4 on Physics and 5 on English. How many different collections can be made such that each collection consists of :

C_1	C_2
(a) One book of each subject;	(i) 3968
(b) At least one book of each subject :	(ii) 60
(c) At least one book of English:	(iii) 3255

61. Five boys and five girls form a line. Find the number of ways of making the seating arrangement under the following condition:

C_1	C_2
(a) Boys and girls alternate:	(i) $5! \times 6!$
(b) No two girls sit together :	(ii) $10! - 5!\,6!$
(c) All the girls sit together	(iii) $(5!)^2 + (5!)^2$
(d) All the girls are never together :	(iv) $2!\,5!\,5!$

62. There are 10 professors and 20 lecturers out of whom a committee of 2 professors and 3 lecturer is to be formed. Find :

	C_1		C_2
(a)	In how many ways committee : can be formed	(i)	${}^{10}C_2 \times {}^{19}C_3$
(b)	In how many ways a particular : professor is included	(ii)	${}^{10}C_2 \times {}^{19}C_2$
(c)	In how many ways a particular : lecturer is included	(iii)	${}^{9}C_1 \times {}^{20}C_3$
(d)	In how many ways a particular : lecturer is excluded	(iv)	${}^{10}C_2 \times {}^{20}C_3$

63. Using the digits 1, 2, 3, 4, 5, 6, 7, a number of 4 different digits is formed. Find

	C_1		C_2
(a)	how many numbers are formed?	(i)	840
(b)	how many numbers are exactly divisible by 2?	(ii)	200
(c)	how many numbers are exactly divisible by 25?	(iii)	360
(d)	how many of these are exactly divisble by 4?	(iv)	40

64. How many words (with or without dictionary meaning) can be made from the letters of the word MONDAY, assuming that no letter is repeated, if

	C_1		C_2
(a)	4 letters are used at a time	(i)	720
(b)	All letters are used at a time	(ii)	240
(c)	All letters are used but the first is a vowel	(iii)	360

Chapter 8

BINOMIAL THEOREM

8.1 Overview:

8.1.1 An expression consisting of two terms, connected by + or – sign is called a binomial expression. For example, $x+a$, $2x-3y$, $\frac{1}{x}-\frac{1}{x^3}$, $7x-\frac{4}{5y}$, etc., are all binomial expressions.

8.1.2 *Binomial theorem*

If a and b are real numbers and n is a positive integer, then

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots$$

$$\ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, b^n, \text{ where } {}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} \text{ for } 0 \le r \le n$$

The general term or $(r+1)^{\text{th}}$ term in the expansion is given by

$$T_{r+1} = {}^nC_r\, a^{n-r} b^r$$

8.1.3 *Some important observations*

1. The total number of terms in the binomial expansion of $(a+b)^n$ is $n+1$, i.e. one more than the exponent n.
2. In the expansion, the first term is raised to the power of the binomial and in each subsequent terms the power of a reduces by one with simultaneous increase in the power of b by one, till power of b becomes equal to the power of binomial, i.e., the power of a is n in the first term, $(n-1)$ in the second term and so on ending with zero in the last term. At the same time power of b is 0 in the first term, 1 in the second term and 2 in the third term and so on, ending with n in the last term.
3. In any term the sum of the indices (exponents) of 'a' and 'b' is equal to n (i.e., the power of the binomial).
4. The coefficients in the expansion follow a certain pattern known as pascal's triangle.

Index of Binomial	Coefficient of various terms
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Each coefficient of any row is obtained by adding two coefficients in the preceding row, one on the immediate left and the other on the immediate right and each row is bounded by 1 on both sides.

The $(r + 1)^{\text{th}}$ term or general term is given by

$$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$$

8.1.4 *Some particular cases*

If n is a positive integer, then

$$(a + b)^n = {}^nC_0\, a^n\, b^0 + {}^nC_1\, a^n\, b^1 + {}^nC_2\, a^{n-2}\, b^2 + ... + {}^nC_r\, a^{n-r}\, b^r + ... + {}^nC_n\, a^0\, b^n \quad ... (1)$$

In particular

1. Replacing b by $-b$ in (i), we get

   $$(a - b)^n = {}^nC_0\, a^n\, b^0 - {}^nC_1\, a^{n-1}\, b^1 + {}^nC_2\, a^{n-2}\, b^2 + ... + (-1)^r\, {}^nC_r\, a^{n-r}\, b^r + ... + (-1)^n\, {}^nC_n\, a^0\, b^n \quad ... (2)$$

2. Adding (1) and (2), we get

   $$(a + b)^n + (a - b)^n = 2\,[{}^nC_0\, a^n\, b^0 + {}^nC_2\, a^{n-2}\, b^2 + {}^nC_4\, a^{n-4}\, b^4 + ...\,]$$
   $$= 2\,[\text{terms at odd places}]$$

3. Subtracting (2) from (1), we get

   $$(a + b)^n - (a - b)^n = 2\,[{}^nC_1\, a^{n-1}\, b^1 + {}^nC_3\, a^{n-3}\, b^3 + ...\,]$$
   $$= 2\,[\text{sum of terms at even places}]$$

4. Replacing a by 1 and b by x in (1), we get

   $$(1 + x)^n = {}^nC_0\, x^0 + {}^nC_1\, x + {}^nC_2\, x^2 + ... + {}^nC_r\, x^r + ... + {}^nC_{n-1}\, x^{n-1} + {}^nC_n\, x^n$$

   i.e. $$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

5. Replacing a by 1 and b by $-x$ in ... (1), we get

$$(1-x)^n = {}^nC_0\, x^0 - {}^nC_1\, x + {}^nC_2\, x^2 \;\ldots + {}^nC_{n-1}\,(-1)^{n-1}\, x^{n-1} + {}^nC_n\,(-1)^n\, x^n$$

i.e., $$(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$$

8.1.5 *The p^{th} term from the end*

The p^{th} term from the end in the expansion of $(a+b)^n$ is $(n-p+2)^{\text{th}}$ term from the beginning.

8.1.6 *Middle terms*

The middle term depends upon the value of n.

(a) If n is even: then the total number of terms in the expansion of $(a+b)^n$ is $n+1$ (odd). Hence, there is only one middle term, i.e., $\left(\frac{n}{2}+1\right)^{\text{th}}$ term is the middle term.

(b) If n is odd: then the total number of terms in the expansion of $(a+b)^n$ is $n+1$ (even). So there are two middle terms i.e., $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ are two middle terms.

8.1.7 *Binomial coefficient*

In the Binomial expression, we have

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}\, b + {}^nC_2\, a^{n-2}\, b^2 + \ldots + {}^nC_n\, b^n \qquad \ldots (1)$$

The coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are known as binomial or combinatorial coefficients.

Putting $a = b = 1$ in (1), we get

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$$

Thus the sum of all the binomial coefficients is equal to 2^n.

Again, putting $a = 1$ and $b = -1$ in (i), we get

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots$$

Thus, the sum of all the odd binomial coefficients is equal to the sum of all the even binomial coefficients and each is equal to $\frac{2^n}{2} = 2^{n-1}$.

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$$

8.2 Solved Examples

Short Answer Type

Example 1 Find the r^{th} term in the expansion of $\left(x+\frac{1}{x}\right)^{2r}$.

Solution We have $T_r = {}^{2r}C_{r-1}\,(x)^{2r-r+1}\left(\frac{1}{x}\right)^{r-1}$

$$= \frac{\lfloor 2r}{\lfloor r-1 \lfloor r+1}\, x^{r+1-r+1}$$

$$= \frac{\lfloor 2r}{\lfloor r-1 \lfloor r+1}\, x^{2}$$

Example 2 Expand the following $(1-x+x^2)^4$

Solution Put $1-x=y$. Then

$$\begin{aligned}(1-x+x^2)^4 &= (y+x^2)^4\\ &= {}^4C_0\, y^4 (x^2)^0 + {}^4C_1\, y^3 (x^2)^1\\ &\quad + {}^4C_2\, y^2 (x^2)^2 + {}^4C_3\, y (x^2)^3 + {}^4C_4\, (x^2)^4\\ &= y^4 + 4y^3x^2 + 6y^2x^4 + 4y\,x^6 + x^8\\ &= (1-x)^4 + 4x^2(1-x)^3 + 6x^4(1-x)^2 + 4x^6(1-x) + x^8\\ &= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8\end{aligned}$$

Example 3 Find the 4^{th} term from the end in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^9$

Solution Since r^{th} term from the end in the expansion of $(a+b)^n$ is $(n-r+2)^{\text{th}}$ term from the beginning. Therefore 4^{th} term from the end is $9-4+2$, i.e., 7^{th} term from the beginning, which is given by

$$T_7 = {}^9C_6\left(\frac{x^3}{2}\right)^3\left(\frac{-2}{x^2}\right)^6 = {}^9C_3\,\frac{x^9}{8}\cdot\frac{64}{x^{12}} = \frac{9\times 8\times 7}{3\times 2\times 1}\times\frac{64}{x^3} = \frac{672}{x^3}$$

Example 4 Evaluate: $\left(x^2-\sqrt{1-x^2}\right)^4 + \left(x^2+\sqrt{1-x^2}\right)^4$

Solution Putting $\sqrt{1-x^2} = y$, we get

The given expression = $(x^2 - y)^4 + (x^2 + y)^4 = 2\ [x^8 + {}^4C_2\ x^4\ y^2 + {}^4C_4\ y^4]$

$$= 2\left[x^8 + \frac{4\times 3}{2\times 1}x^4\cdot(1-x^2) + (1-x^2)^2\right]$$

$$= 2\ [x^8 + 6x^4\ (1 - x^2) + (1 - 2x^2 + x^4]$$

$$= 2x^8 - 12x^6 + 14x^4 - 4x^2 + 2$$

Example 5 Find the coefficient of x^{11} in the expansion of $\left(x^3 - \frac{2}{x^2}\right)^{12}$

Solution Let the general term, i.e., $(r + 1)^{th}$ contain x^{11}.

We have $\qquad T_{r+1} = {}^{12}C_r\ (x^3)^{12-r}\left(-\frac{2}{x^2}\right)^r$

$$= {}^{12}C_r\ x^{36-3r-2r}\ (-1)^r\ 2^r$$

$$= {}^{12}C_r\ (-1)^r\ 2^r\ x^{36-5r}$$

Now for this to contain x^{11}, we observe that

$$36 - 5r = 11, \text{ i.e., } r = 5$$

Thus, the coefficient of x^{11} is

$$^{12}C_5\ (-1)^5\ 2^5 = -\frac{12\times 11\times 10\times 9\times 8}{5\times 4\times 3\times 2}\times 32 = -25344$$

Example 6 Determine whether the expansion of $\left(x^2 - \frac{2}{x}\right)^{18}$ will contain a term containing x^{10}?

Solution Let T_{r+1} contain x^{10}. Then

$$T_{r+1} = {}^{18}C_r\,(x^2)^{18-r}\left(\frac{-2}{x}\right)^r$$

$$= {}^{18}C_r\ x^{36-2r}\ (-1)^r\ .\ 2^r\ x^{-r}$$

$$= (-1)^r\ 2^r\ {}^{18}C_r\ x^{36-3r}$$

Thus, $\qquad 36 - 3r = 10$, i.e., $r = \frac{26}{3}$

Since r is a fraction, the given expansion cannot have a term containing x^{10}.

Example 7 Find the term independent of x in the expansion of $\left(\frac{\sqrt{x}}{\sqrt{3}}+\frac{\sqrt{3}}{2x^2}\right)^{10}$.

Solution Let $(r + 1)^{\text{th}}$ term be independent of x which is given by

$$T_{r+1} = {}^{10}C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \left(\frac{\sqrt{3}}{2x^2}\right)^r$$

$$= {}^{10}C_r \left(\frac{x}{3}\right)^{\frac{10-r}{2}} 3^{\frac{r}{2}} \left(\frac{1}{2^r x^{2r}}\right)$$

$$= {}^{10}C_r\ 3^{\frac{r}{2}-\frac{10-r}{2}} 2^{-r} x^{\frac{10-r}{2}-2r}$$

Since the term is independent of x, we have

$$\frac{10-r}{2}-2r = 0 \qquad \Rightarrow \qquad r = 2$$

Hence 3rd term is independent of x and its value is given by

$$T_3 = {}^{10}C_2 \frac{3^{-3}}{4} = \frac{10\times 9}{2\times 1}\times\frac{1}{9\times 12} = \frac{5}{12}$$

Example 8 Find the middle term in the expansion of $\left(2ax-\frac{b}{x^2}\right)^{12}$.

Solution Since the power of binomial is even, it has one middle term which is the $\left(\frac{12+2}{2}\right)^{\text{th}}$ term and it is given by

$$T_7 = {}^{12}C_6 (2ax)^6 \left(\frac{-b}{x^2}\right)^6$$

$$= {}^{12}C_6 \frac{2^6 a^6 x^6 \cdot (-b)^6}{x^{12}}$$

$$= {}^{12}C_6 \frac{2^6 a^6 b^6}{x^6} = \frac{59136 a^6 b^6}{x^6}$$

Example 9 Find the middle term (terms) in the expansion of $\left(\frac{p}{x}+\frac{x}{p}\right)^9$.

Solution Since the power of binomial is odd. Therefore, we have two middle terms which are 5th and 6th terms. These are given by

$$T_5 = {}^9C_4\left(\frac{p}{x}\right)^5\left(\frac{x}{p}\right)^4 = {}^9C_4\frac{p}{x} = \frac{126p}{x}$$

and $$T_6 = {}^9C_5\left(\frac{p}{x}\right)^4\left(\frac{x}{p}\right)^5 = {}^9C_5\frac{x}{p} = \frac{126x}{p}$$

Example 10 Show that $2^{4n+4} - 15n - 16$, where $n \in \mathbf{N}$ is divisible by 225.

Solution We have

$$\begin{aligned}
2^{4n+4} - 15n - 16 &= 2^{4(n+1)} - 15n - 16 \\
&= 16^{n+1} - 15n - 16 \\
&= (1+15)^{n+1} - 15n - 16 \\
&= {}^{n+1}C_0\, 15^0 + {}^{n+1}C_1\, 15^1 + {}^{n+1}C_2\, 15^2 + {}^{n+1}C_3\, 15^3 \\
&\quad + \ldots + {}^{n+1}C_{n+1}\,(15)^{n+1} - 15n - 16 \\
&= 1 + (n+1)\,15 + {}^{n+1}C_2\, 15^2 + {}^{n+1}C_3\, 15^3 \\
&\quad + \ldots + {}^{n+1}C_{n+1}\,(15)^{n+1} - 15n - 16 \\
&= 1 + 15n + 15 + {}^{n+1}C_2\, 15^2 + {}^{n+1}C_3\, 15^3 \\
&\quad + \ldots + {}^{n+1}C_{n+1}\,(15)^{n+1} - 15n - 16 \\
&= 15^2\,[{}^{n+1}C_2 + {}^{n+1}C_3\, 15 + \ldots \text{ so on}]
\end{aligned}$$

Thus, $2^{4n+4} - 15n - 16$ is divisible by 225.

Long Answer Type

Example 11 Find numerically the greatest term in the expansion of $(2 + 3x)^9$, where $x = \frac{3}{2}$.

Solution We have $(2 + 3x)^9 = 2^9\left(1+\frac{3x}{2}\right)^9$

Now,
$$\frac{T_{r+1}}{T_r} = \frac{2^9\left[{}^9C_r\left(\frac{3x}{2}\right)^r\right]}{2^9\left[{}^9C_{r-1}\left(\frac{3x}{2}\right)^{r-1}\right]}$$

$$= \frac{{}^9C_r}{{}^9C_{r-1}}\left|\frac{3x}{2}\right| = \frac{\underline{|9}}{\underline{|r}\,\underline{|9-r}} \cdot \frac{\underline{|r-1}\,\underline{|10-r}}{\underline{|9}}\left|\frac{3x}{2}\right|$$

$$= \frac{10-r}{r}\left|\frac{3x}{2}\right| = \frac{10-r}{r}\left(\frac{9}{4}\right) \qquad \text{Since} \quad x = \frac{3}{2}$$

Therefore,
$$\frac{T_{r+1}}{T_r} \ge 1 \Rightarrow \frac{90-9r}{4r} \ge 1$$

$$\Rightarrow 90 - 9r \ge 4r \qquad \text{(Why?)}$$

$$\Rightarrow r \le \frac{90}{13}$$

$$\Rightarrow r \le 6\,\frac{12}{13}$$

Thus the maximum value of r is 6. Therefore, the greatest term is $T_{r+1} = T_7$.

Hence,
$$T_7 = 2^9\left[{}^9C_6\left(\frac{3x}{2}\right)^6\right], \qquad \text{where } x = \frac{3}{2}$$

$$= 2^9 \cdot {}^9C_6\left(\frac{9}{4}\right)^6 = 2^9 \cdot \frac{9\times8\times7}{3\times2\times1}\left(\frac{3^{12}}{2^{12}}\right) = \frac{7\times3^{13}}{2}$$

Example 12 If n is a positive integer, find the coefficient of x^{-1} in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$.

Solution We have

$$(1+x)^n\left(1+\frac{1}{x}\right)^n = (1+x)^n\left(\frac{x+1}{x}\right)^n = \frac{(1+x)^{2n}}{x^n}$$

Now to find the coefficient of x^{-1} in $(1 + x)^n \left(1+\frac{1}{x}\right)^n$, it is equivalent to finding coefficient of x^{-1} in $\frac{(1+x)^{2n}}{x^n}$ which in turn is equal to the coefficient of x^{n-1} in the expansion of $(1 + x)^{2n}$.

Since $(1 + x)^{2n} = {}^{2n}C_0\, x^0 + {}^{2n}C_1\, x^1 + {}^{2n}C_2\, x^2 + \ldots + {}^{2n}C_{n-1}\, x^{n-1} + \ldots + {}^{2n}C_{2n}\, x^{2n}$

Thus the coefficient of x^{n-1} is ${}^{2n}C_{n-1}$

$$= \frac{\lfloor 2n}{\lfloor n-1 \lfloor 2n-n+1} = \frac{\lfloor 2n}{\lfloor n-1 \lfloor n+1}$$

Example 13 Which of the following is larger?

$99^{50} + 100^{50}$ or 101^{50}

We have $(101)^{50} = (100 + 1)^{50}$

$$= 100^{50} + 50\,(100)^{49} + \frac{50.49}{2.1}(100)^{48} + \frac{50.49.48}{3.2.1}(100)^{47} + \ldots \quad (1)$$

Similarly $99^{50} = (100 - 1)^{50}$

$$= 100^{50} - 50\,.\,100^{49} + \frac{50.49}{2.1}(100)^{48} - \frac{50.49.48}{3.2.1}(100)^{47} + \ldots \quad (2)$$

Subtracting (2) from (1), we get

$$101^{50} - 99^{50} = 2\left[50\cdot(100)^{49} + \frac{50\cdot 49\cdot 48}{3\cdot 2\cdot 1}100^{47} + \ldots\right]$$

$$\Rightarrow \quad 101^{50} - 99^{50} = 100^{50} + 2\left(\frac{50\cdot 49\cdot 48}{3\cdot 2\cdot 1}\right)100^{47} + \ldots$$

$$\Rightarrow \quad 101^{50} - 99^{50} > 100^{50}$$

Hence $101^{50} > 99^{50} + 100^{50}$

Example 14 Find the coefficient of x^{50} after simplifying and collecting the like terms in the expansion of $(1 + x)^{1000} + x\,(1 + x)^{999} + x^2\,(1 + x)^{998} + \ldots + x^{1000}$.

Solution Since the above series is a geometric series with the common ratio $\frac{x}{1+x}$, its sum is

$$\frac{(1+x)^{1000}\left[1-\left(\frac{x}{1+x}\right)^{1001}\right]}{\left[1-\left(\frac{x}{1+x}\right)\right]}$$

$$= \frac{(1+x)^{1000} - \frac{x^{1001}}{1+x}}{\frac{1+x-x}{1+x}} = (1+x)^{1001} - x^{1001}$$

Hence, coefficient of x^{50} is given by

$$^{1001}C_{50} = \frac{\lfloor 1001}{\lfloor 50 \lfloor 951}$$

Example 15 If a_1, a_2, a_3 and a_4 are the coefficient of any four consecutive terms in the expansion of $(1 + x)^n$, prove that

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$$

Solution Let a_1, a_2, a_3 and a_4 be the coefficient of four consecutive terms T_{r+1}, T_{r+2}, T_{r+3}, and T_{r+4} respectively. Then

a_1 = coefficient of $T_{r+1} = {}^nC_r$

a_2 = coefficient of $T_{r+2} = {}^nC_{r+1}$

a_3 = coefficient of $T_{r+3} = {}^nC_{r+2}$

and a_4 = coefficient of $T_{r+4} = {}^nC_{r+3}$

Thus $$\frac{a_1}{a_1+a_2} = \frac{^nC_r}{^nC_r + {}^nC_{r+1}}$$

$$= \frac{^nC_r}{^{n+1}C_{r+1}} \qquad (\because \ {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$$

$$= \frac{\lfloor n}{\lfloor r \lfloor n-r} \times \frac{\lfloor r+1 \lfloor n-r}{\lfloor n+1} = \left(\frac{r+1}{n+1}\right)$$

Similarly, $$\frac{a_3}{a_3 + a_4} = \frac{{}^nC_{r+2}}{{}^nC_{r+2} + {}^nC_{r+3}}$$

$$= \frac{{}^nC_{r+2}}{{}^{n+1}C_{r+3}} = \frac{r+3}{n+1}$$

Hence, $$\text{L.H.S.} = \frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{r+1}{n+1} + \frac{r+3}{n+1} = \frac{2r+4}{n+1}$$

and $$\text{R.H.S.} = \frac{2a_2}{a_2 + a_3} = \frac{2\left({}^nC_{r+1}\right)}{{}^nC_{r+1} + {}^nC_{r+2}} = \frac{2\left({}^nC_{r+1}\right)}{{}^{n+1}C_{r+2}}$$

$$= 2\frac{\lfloor n}{\lfloor r+1 \lfloor n-r-1} \times \frac{\lfloor r+2 \lfloor n-r-1}{\lfloor n+1} = \frac{2\,(r+2)}{n+1}$$

Objective Type Questions (M.C.Q)

Example 16 The total number of terms in the expansion of $(x + a)^{51} - (x - a)^{51}$ after simplification is

(a) 102 (b) 25 (c) 26 (d) None of these

Solution C is the correct choice since the total number of terms are 52 of which 26 terms get cancelled.

Example 17 If the coefficients of x^7 and x^8 in $\left(2 + \frac{x}{3}\right)^n$ are equal, then n is

(a) 56 (b) 55 (c) 45 (d) 15

Solution B is the correct choice. Since $T_{r+1} = {}^nC_r\, a^{n-r} x^r$ in expansion of $(a + x)^n$,

Therefore, $$T_8 = {}^nC_7 (2)^{n-7} \left(\frac{x}{3}\right)^7 = {}^nC_7 \frac{2^{n-7}}{3^7} x^7$$

and $$T_9 = {}^nC_8 (2)^{n-8} \left(\frac{x}{3}\right)^8 = {}^nC_8 \frac{2^{n-8}}{3^8} x^8$$

Therefore, $${}^nC_7 \frac{2^{n-7}}{3^7} = {}^nC_8 \frac{2^{n-8}}{3^8}$$ (since it is given that coefficient of x^7 = coefficient x^8)

$$\Rightarrow \quad \frac{\lfloor n}{\lfloor 7 \lfloor n-7} \times \frac{\lfloor 8 \lfloor n-8}{\lfloor n} = \frac{2^{n-8}}{3^8} \cdot \frac{3^7}{2^{n-7}}$$

$$\Rightarrow \quad \frac{8}{n-7} = \frac{1}{6} \quad \Rightarrow \quad n = 55$$

Example 18 If $(1 - x + x^2)^n = a_0 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$, then $a_0 + a_2 + a_4 + ... + a_{2n}$ equals.

(A) $\frac{3^n + 1}{2}$ (B) $\frac{3^n - 1}{2}$ (C) $\frac{1 - 3^n}{2}$ (D) $3^n + \frac{1}{2}$

Solution A is the correct choice. Putting $x = 1$ and -1 in

$$(1 - x + x^2)^n = a_0 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$$

we get $$1 = a_0 + a_1 + a_2 + a_3 + ... + a_{2n} \quad ... (1)$$

and $$3^n = a_0 - a_1 + a_2 - a_3 + ... + a_{2n} \quad ... (2)$$

Adding (1) and (2), we get

$$3^n + 1 = 2 (a_0 + a_2 + a_4 + ... + a_{2n})$$

Therefore $$a_0 + a_2 + a_4 + ... + a_{2n} = \frac{3^n + 1}{2}$$

Example 19 The coefficient of x^p and x^q (p and q are positive integers) in the expansion of $(1 + x)^{p+q}$ are

(A) equal (B) equal with opposite signs

(C) reciprocal of each other (D) none of these

Solution A is the correct choice. Coefficient of x^p and x^q in the expansion of $(1 + x)^{p+q}$ are ${}^{p+q}C_p$ and ${}^{p+q}C_q$

and $${}^{p+q}C_p = {}^{p+q}C_q = \frac{\lfloor p+q}{\lfloor p \lfloor q}$$

Hence (a) is the correct answer.

Example 20 The number of terms in the expansion of $(a + b + c)^n$, where $n \in \mathbf{N}$ is

(A) $\dfrac{(n+1)(n+2)}{2}$ (B) $n + 1$

(C) $n + 2$ (D) $(n + 1)\, n$

Solution A is the correct choice. We have

$$\begin{aligned}(a + b + c)^n &= [a + (b + c)]^n \\ &= a^n + {}^nC_1\, a^{n-1} (b + c)^1 + {}^nC_2\, a^{n-2} (b + c)^2 \\ &\quad + ... + {}^nC_n (b + c)^n\end{aligned}$$

Further, expanding each term of R.H.S., we note that

First term consist of 1 term.

Second term on simplification gives 2 terms.

Third term on expansion gives 3 terms.

Similarly, fourth term on expansion gives 4 terms and so on.

$$\text{The total number of terms} = 1 + 2 + 3 + ... + (n + 1)$$

$$= \frac{(n+1)(n+2)}{2}$$

Example 21 The ratio of the coefficient of x^{15} to the term independent of x in $\left(x^2 + \dfrac{2}{x}\right)^{15}$ is

(A) 12:32 (B) 1:32 (C) 32:12 (D) 32:1

Solution (B) is the correct choice. Let T_{r+1} be the general term of $\left(x^2 + \dfrac{2}{x}\right)^{15}$, so,

$$T_{r+1} = {}^{15}C_r\, (x^2)^{15-r} \left(\frac{2}{x}\right)^r$$

$$= {}^{15}C_r\, (2)^r\, x^{30-3r} \qquad ... (1)$$

Now, for the coefficient of term containing x^{15},

$$30 - 3r = 15, \quad \text{i.e.,} \quad r = 5$$

Therefore, ${}^{15}C_5\, (2)^5$ is the coefficient of x^{15} (from (1))

To find the term independent of x, put $30 - 3r = 0$

Thus ${}^{15}C_{10}\ 2^{10}$ is the term independent of x (from (1))

Now the ratio is $\dfrac{{}^{15}C_5\ 2^5}{{}^{15}C_{10}\ 2^{10}} = \dfrac{1}{2^5} = \dfrac{1}{32}$

Example 22 If $z = \left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5$, then

(A) Re $(z) = 0$ (B) $I_m\ (z) = 0$

(C) Re $(z) > 0$, $I_m\ (z) > 0$ (D) Re $(z) > 0$, $I_m\ (z) < 0$

Solution B is the correct choice. On simplification, we get

$$z = 2\left[{}^5C_0\left(\frac{\sqrt{3}}{2}\right)^2 + {}^5C_2\left(\frac{\sqrt{3}}{2}\right)^3\left(\frac{i}{2}\right)^2 + {}^5C_4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{i}{2}\right)^4\right]$$

Since $i^2 = -1$ and $i^4 = 1$, z will not contain any i and hence $I_m\ (z) = 0$.

8.3 EXERCISE

Short Answer Type

1. Find the term independent of x, $x \neq 0$, in the expansion of $\left(\dfrac{3x^2}{2} - \dfrac{1}{3x}\right)^{15}$.

2. If the term free from x in the expansion of $\left(\sqrt{x} - \dfrac{k}{x^2}\right)^{10}$ is 405, find the value of k.

3. Find the coefficient of x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$.

4. Find the term independent of x in the expansion of, $\left(3x - \dfrac{2}{x^2}\right)^{15}$.

5. Find the middle term (terms) in the expansion of

   (i) $\left(\dfrac{x}{a} - \dfrac{a}{x}\right)^{10}$ (ii) $\left(3x - \dfrac{x^3}{6}\right)^9$

6. Find the coefficient of x^{15} in the expansion of $(x - x^2)^{10}$.

7. Find the coefficient of $\frac{1}{x^{17}}$ in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

8. Find the sixth term of the expansion $\left(y^{\frac{1}{2}} + x^{\frac{1}{3}}\right)^n$, if the binomial coefficient of the third term from the end is 45.

[**Hint:** Binomial coefficient of third term from the end = Binomial coefficient of third term from beginning = ${}^nC_2$.]

9. Find the value of r, if the coefficients of $(2r + 4)^{\text{th}}$ and $(r - 2)^{\text{th}}$ terms in the expansion of $(1 + x)^{18}$ are equal.

10. If the coefficient of second, third and fourth terms in the expansion of $(1 + x)^{2n}$ are in A.P. Show that $2n^2 - 9n + 7 = 0$.

11. Find the coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$.

Long Answer Type

12. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2} + 2\right)^8$ is 1120, find p.

13. Show that the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is

$$\frac{1 \times 3 \times 5 \times \ldots (2n-1)}{\lfloor n} \times (-2)^n.$$

14. Find n in the binomial $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ if the ratio of 7^{th} term from the beginning to the 7^{th} term from the end is $\frac{1}{6}$.

15. In the expansion of $(x + a)^n$ if the sum of odd terms is denoted by O and the sum of even term by E.

Then prove that

(i) $O^2 - E^2 = (x^2 - a^2)^n$ (ii) $4OE = (x + a)^{2n} - (x - a)^{2n}$

16. If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is

$$\frac{\lfloor 2n}{\lfloor \frac{4n-p}{3} \lfloor \frac{2n+p}{3}}.$$

17. Find the term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

Objective Type Questions

Choose the correct answer from the given options in each of the Exercises 18 to 24 (M.C.Q.).

18. The total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification is

(A) 50 (B) 202 (C) 51 (D) none of these

19. Given the integers $r > 1$, $n > 2$, and coefficients of $(3r)^{\text{th}}$ and $(r + 2)^{\text{nd}}$ terms in the binomial expansion of $(1 + x)^{2n}$ are equal, then

(A) $n = 2r$ (B) $n = 3r$ (C) $n = 2r + 1$ (D) none of these

20. The two successive terms in the expansion of $(1 + x)^{24}$ whose coefficients are in the ratio 1:4 are

(A) 3^{rd} and 4^{th} (B) 4^{th} and 5^{th} (C) 5^{th} and 6^{th} (D) 6^{th} and 7^{th}

[**Hint**: $\frac{{}^{24}C_r}{{}^{24}C_{r+1}} = \frac{1}{4} \Rightarrow \frac{r+1}{24-r} = \frac{1}{4} \Rightarrow 4r + 4 = 24 - 4 \Rightarrow \boxed{r = 4}$]

21. The coefficient of x^n in the expansion of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ are in the ratio.

(A) 1 : 2 (B) 1 : 3 (C) 3 : 1 (D) 2 : 1

[**Hint** : ${}^{2n}C_n : {}^{2n-1}C_n$

22. If the coefficients of 2^{nd}, 3^{rd} and the 4^{th} terms in the expansion of $(1 + x)^n$ are in A.P., then value of n is

(A) 2 (B) 7 (c) 11 (D) 14

[**Hint:** $2\ {}^nC_2 = {}^nC_1 + {}^nC_3 \Rightarrow n^2 - 9n + 14 = 0 \Rightarrow n = 2$ or 7

23. If A and B are coefficient of x^n in the expansions of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then $\frac{A}{B}$ equals

(A) 1 (B) 2 (C) $\frac{1}{2}$ (D) $\frac{1}{n}$

[**Hint:** $\frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = 2$]

24. If the middle term of $\left(\frac{1}{x} + x\sin x\right)^{10}$ is equal to $7\frac{7}{8}$, then value of x is

(A) $2n\pi + \frac{\pi}{6}$ (B) $n\pi + \frac{\pi}{6}$ (C) $n\pi + (-1)^n \frac{\pi}{6}$ (D) $n\pi + (-1)^n \frac{\pi}{3}$

[**Hint:** $T_6 = {}^{10}C_5 \frac{1}{x^5} \cdot x^5 \sin^5 x = \frac{63}{8} \Rightarrow \sin^5 x = \frac{1}{2^5} \Rightarrow \sin = \frac{1}{2}$

$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}$]

Fill in the blanks in Exercises 25 to 33.

25. The largest coefficient in the expansion of $(1 + x)^{30}$ is ________________ .

26. The number of terms in the expansion of $(x + y + z)^n$ ________________ .

[**Hint:** $(x + y + z)^n = [x + (y + z)]^n$]

27. In the expansion of $\left(x^2 - \frac{1}{x^2}\right)^{16}$, the value of constant term is ________________ .

28. If the seventh terms from the beginning and the end in the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ are equal, then n equals ________________ .

[**Hint :** $T_7 = T_{n-7+2} \Rightarrow {}^nC_6 \left(2^{\frac{1}{3}}\right)^{n-6} \left(\frac{1}{3^{\frac{1}{3}}}\right)^6 = {}^nC_{n-6} \left(2^{\frac{1}{3}}\right)^6 \left(\frac{1}{3^{\frac{1}{3}}}\right)^{n-6}$

$$\Rightarrow \left(2^{\frac{1}{3}}\right)^{n-12} = \left(\frac{1}{3^{\frac{1}{3}}}\right)^{n-12} \Rightarrow \text{only problem when } n - 12 = 0 \Rightarrow n = 12].$$

29. The coefficient of $a^{-6} b^4$ in the expansion of $\left(\frac{1}{a} - \frac{2b}{3}\right)^{10}$ is __________.

[**Hint :** $T_5 = {}^{10}C_4 \left(\frac{1}{a}\right)^b \left(\frac{-2b}{3}\right)^4 = \frac{1120}{27} a^{-6} b^4$]

30. Middle term in the expansion of $(a^3 + ba)^{28}$ is __________ .

31. The ratio of the coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ is __________

[**Hint:** ${}^{p+q}C_p = {}^{p+q}C_q$]

$\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$ is

__________ .

33. If 25^{15} is divided by 13, the reminder is __________ .

State which of the statement in Exercises 34 to 40 is True or False.

34. The sum of the series $\sum_{r=0}^{10} {}^{20}C_r$ is $2^{19} + \frac{{}^{20}C_{10}}{2}$

35. The expression $7^9 + 9^7$ is divisible by 64.

Hint: $7^9 + 9^7 = (1 + 8)^7 - (1 - 8)^9$

36. The number of terms in the expansion of $[(2x + y^3)^4]^7$ is 8 **37.**

The sum of coefficients of the two middle terms in the expansion of $(1 + x)^{2n-1}$ is equal to ${}^{2n-1}C_n$.

38. The last two digits of the numbers 3^{400} are 01.

39. If the expansion of $\left(x - \frac{1}{x^2}\right)^{2n}$ contains a term independent of x, then n is a multiple of 2.

40. Number of terms in the expansion of $(a + b)^n$ where $n \in \mathbf{N}$ is one less than the power n.

Chapter 9

SEQUENCE AND SERIES

9.1 Overview

By a sequence, we mean an arrangement of numbers in a definite order according to some rule. We denote the terms of a sequence by $a_1, a_2, a_3, \ldots$, etc., the subscript denotes the position of the term.

In view of the above a sequence in the set X can be regarded as a mapping or a function $f: \mathbf{N} \rightarrow X$ defined by

$$f(n) = t_n \ \forall \ n \in \mathbf{N}.$$

Domain of f is a set of natural numbers or some subset of it denoting the position of term. If its range denoting the value of terms is a subset of **R** real numbers then it is called a **real sequence**.

A sequence is either finite or infinite depending upon the number of terms in a sequence. We should not expect that its terms will be necessarily given by a specific formula.

However, we expect a theoretical scheme or rule for generating the terms.

Let $a_1, a_2, a_3, \ldots$, be the sequence, then, the expression $a_1 + a_2 + a_3 + \ldots$ is called the **series** associated with given sequence. The series is finite or infinite according as the given sequence is finite or infinite.

Remark When the series is used, it refers to the indicated sum not to the sum itself. Sequence following certain patterns are more often called **progressions**. In progressions, we note that each term except the first progresses in a definite manner.

9.1.1 ***Arithmetic progression (A.P.)*** is a sequence in which each term except the first is obtained by adding a fixed number (positive or negative) to the preceding term.

Thus any sequence $a_1, a_2, a_3 \ldots a_n, \ldots$ is called an **arithmetic progression** if

$a_{n+1} = a_n + d, n \in \mathbf{N}$, where d is called the **common difference** of the A.P., usually we denote the first term of an A.P by a and the last term by l

The general term or the **n^{th} term of the A.P.** is given by

$$\boldsymbol{a_n = a + (n-1)\,d}$$

The n^{th} term from the last is given by $\quad a_n = l - (n-1)\,d$

The sum S_n of the first n terms of an A.P. is given by

$S_n = \frac{n}{2}\left[2a + (n-1)\,d\right] = \frac{n}{2}(a + l)$, where $l = a + (n-1)\,d$ is the last terms of the A.P.,

and the general term is given by $a_n = S_n - S_{n-1}$

The arithmetic mean for any n positive numbers $a_1, a_2, a_3, \ldots a_n$ is given by

$$\text{A.M.} = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

If a, A and b are in A.P., then A is called the arithmetic mean of numbers a and b and

i.e., $$A = \frac{a+b}{2}$$

If the terms of an A.P. are increased, decreased, multiplied or divided by the same constant, they still remain in A.P.

If $a_1, a_2, a_3 \ldots$ are in A.P. with common difference d, then

(i) $a_1 \pm k, a_2 \pm k, a_3 \pm k, \ldots$ are also in A.P with common difference d.

(ii) $a_1 k, a_2 k, a_3 k, \ldots$ are also in A.P with common difference dk $(k \neq 0)$.

and $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}$ ... are also in A.P. with common difference $\frac{d}{k}$ $(k \neq 0)$.

If $a_1, a_2, a_3 \ldots$ and $b_1, b_2, b_3 \ldots$ are two A.P., then

(i) $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are also in A.P

(ii) $a_1 b_1, a_2 b_2, a_3 b_3, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are not in A.P.

If $a_1, a_2, a_3 \ldots$ and a_n are in A.Ps, then

(i) $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$

(ii) $a_r = \frac{a_{r-k} + a_{r+k}}{2} \quad \forall \; k, 0 \leq k \leq n - r$

(iii) If n^{th} term of any sequence is linear expression in n, then the sequence is an A.P.

(iv) If sum of n terms of any sequence is a quadratic expression in n, then sequence is an A.P.

9.1.2 ***A Geometric progression (G.P.)*** is a sequence in which each term except the first is obtained by multiplying the previous term by a non-zero constant called the **common ratio**. Let us consider a G.P. with first non-zero term a and common ratio r, i.e.,

$$a, ar, ar^2, \ldots, ar^{n-1}, \ldots$$

Here, common ratio $r = \dfrac{ar^{n-1}}{ar^{n-2}}$

The **general term** or $\boldsymbol{n}^{\text{th}}$ **term** of G.P. is given by $a_n = ar^{n-1}$.

Last term l of a G.P. is same as the n^{th} term and is given by $l = ar^{n-1}$.

and the n^{th} term from the last is given by $a_n = \dfrac{l}{r^{n-1}}$

The sum S_n of the first n terms is given by

$$S_n = \frac{a(r^n - 1)}{r - 1}, \qquad \text{if } r \neq 1$$

$$S_n = na \qquad \text{if } r = 1$$

If a, G and b are in G.P., then G is called the **geometric mean** of the numbers a and b and is given by

$$G = \sqrt{ab}$$

(i) If the terms of a G.P. are multiplied or divided by the same non-zero constant ($k \neq 0$), they still remain in G.P.

If $a_1, a_2, a_3, \ldots$, are in G.P., then $a_1 k, a_2 k, a_3 k, \ldots$ and $\dfrac{a_1}{k}, \dfrac{a_2}{k}, \dfrac{a_3}{k}, \ldots$

are also in G.P. with same common ratio, in particularly

if $a_1, a_2, a_3, \ldots$ are in G.P., then

$\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \ldots$ are also in G.P.

(ii) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two G.P.s, then $a_1 b_1, a_2 b_2, a_3 b_3, \ldots$ and

$\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2}, \dfrac{a_3}{b_3}, \ldots$ are also in G.P.

(iii) If $a_1, a_2, a_3, \ldots$ are in A.P. ($a_i > 0 \ \forall\ i$), then $x^{a_1}, x^{a_2}, x^{a_3}, \ldots$, are in G.P. ($\forall\ x > 0$)

(iv) If $a_1, a_2, a_3, ..., a_n$ are in G.P., then $a_1\, a_n = a_2\, a_{n-1} = a_3\, a_{n-2} = ...$

9.1.3 *Important results on the sum of special sequences*

(i) Sum of the first n natural numbers:

$$\sum n = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

(ii) Sum of the squares of first n natural numbers.

$$\sum n^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(iii) Sum of cubes of first n natural numbers:

$$\sum n^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

9.2 Solved Examples

Short Answer Type

Example 1 The first term of an A.P. is a, the second term is b and the last term is c. Show that the sum of the A.P. is $\frac{(b+c-2a)(c+a)}{2(b-a)}$.

Solution Let d be the common diffrence and n be the number of terms of the A.P. Since the first term is a and the second term is b

Therefore, $d = b - a$

Also, the last term is c, so

$$c = a + (n-1)(b - \text{a}) \quad (\text{since } d = b - a)$$

$$\Rightarrow \quad n - 1 = \frac{c-a}{b-a}$$

$$\Rightarrow \quad n = 1 + \frac{c-a}{b-a} = \frac{b-a+c-a}{b-a} = \frac{b+c-2a}{b-a}$$

Therefore, $S_n = \frac{n}{2}(a+l) = \frac{(b+c-2a)}{2(b-a)}(a+c)$

Example 2 The p^{th} term of an A.P. is a and q^{th} term is b. Prove that the sum of its $(p + q)$ terms is

$\dfrac{p+q}{2}\left[a+b+\dfrac{a-b}{p-q}\right]$.

Solution Let A be the first term and D be the common difference of the A.P. It is given that

$$t_p = a \Rightarrow \text{A} + (p-1)\,\text{D} = a \qquad \text{... (1)}$$

$$t_q = b \Rightarrow \text{A} + (q-1)\,\text{D} = b \qquad \text{... (2)}$$

Subtracting (2) from (1), we get

$$(p-1-q+1)\,\text{D} = a-b$$

$$\Rightarrow \qquad \text{D} = \frac{a-b}{p-q} \qquad \text{... (3)}$$

Adding (1) and (2), we get

$$2\text{A} + (p+q-2)\,\text{D} = a+b$$

$$\Rightarrow \qquad 2\text{A} + (p+q-1)\,\text{D} = a+b+\text{D}$$

$$\Rightarrow \qquad 2\text{A} + (p+q-1)\,\text{D} = a+b+\frac{a-b}{p-q} \qquad \text{.. (4)}$$

Now
$$\text{S}_{p+q} = \frac{p+q}{2}\,[2\text{A} + (p+q-1)\,\text{D}]$$

$$= \frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$$

[(using ... (3) and (4)]

Example 3 If there are $(2n + 1)$ terms in an A.P., then prove that the ratio of the sum of odd terms and the sum of even terms is $(n + 1) : n$

Solution Let a be the first term and d the common difference of the A.P. Also let S_1 be the sum of odd terms of A.P. having $(2n + 1)$ terms. Then

$$\text{S}_1 = a_1 + a_3 + a_5 + ... + a_{2n+1}$$

$$\text{S}_1 = \frac{n+1}{2}(a_1 + a_{2n+1})$$

$$\text{S}_1 = \frac{n+1}{2}\left[a + a + (2n+1-1)d\right]$$

$$= (n + 1)(a + nd)$$

Similarly, if S_2 denotes the sum of even terms, then

$$S_2 = \frac{n}{2}[2a + 2nd] = n(a + nd)$$

Hence $$\frac{S_1}{S_2} = \frac{(n+1)(a+nd)}{n(a+nd)} = \frac{n+1}{n}$$

Example 4 At the end of each year the value of a certain machine has depreciated by 20% of its value at the beginning of that year. If its initial value was Rs 1250, find the value at the end of 5 years.

Solution After each year the value of the machine is 80% of its value the previous year so at the end of 5 years the machine will depreciate as many times as 5.

Hence, we have to find the 6^{th} term of the G.P. whose first term a_1 is 1250 and common ratio r is .8.

Hence, value at the end 5 years $= t_6 = a_1 r^5 = 1250\,(.8)^5 = 409.6$

Example 5 Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots$ if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

Solution We know that in an A.P., the sum of the terms equidistant from the beginning and end is always the same and is equal to the sum of first and last term.

Therefore $$d = b - a$$

i.e., $$a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$$

It is given that $(a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$

$\Rightarrow (a_1 + a_{24}) + (a_1 + a_{24}) + (a_1 + a_{24}) = 225$

$\Rightarrow$ $$3(a_1 + a_{24}) = 225$$

$\Rightarrow$ $$a_1 + a_{24} = 75$$

We know that $S_n = \frac{n}{2}[a + l]$, where a is the first term and l is the last term of an A.P.

Thus, $$S_{24} = \frac{24}{2}[a_1 + a_{24}] = 12 \times 75 = 900$$

Example 6 The product of three numbers in A.P. is 224, and the largest number is 7 times the smallest. Find the numbers.

Solution Let the three numbers in A.P. be $a - d, a, a + d$ $(d > 0)$

Now $\quad (a - d)\, a\, (a + d) = 224$

$\Rightarrow \quad a\, (a^2 - d^2) = 224 \quad \ldots (1)$

Now, since the largest number is 7 times the smallest, i.e., $a + d = 7\,(a - d)$

Therefore, $\quad d = \dfrac{3a}{4}$

Substituting this value of d in (1), we get

$$a\left(a^2 - \frac{9a^2}{16}\right) = 224$$

$$a = 8$$

and $\quad d = \dfrac{3a}{4} = \dfrac{3}{4} \times 8 = 6$

Hence, the three numbers are 2, 8, 14.

Example 7 Show that $(x^2 + xy + y^2)$, $(z^2 + xz + x^2)$ and $(y^2 + yz + z^2)$ are consecutive terms of an A.P., if x, y and z are in A.P.

Solution The terms $(x^2 + xy + y^2)$, $(z^2 + xz + x^2)$ and $(y^2 + yz + z^2)$ will be in A.P. if

$$(z^2 + xz + x^2) - (x^2 + xy + y^2) = (y^2 + yz + z^2) - (z^2 + xz + x^2)$$

i.e., $\quad z^2 + xz - xy - y^2 = y^2 + yz - xz - x^2$

i.e., $\quad x^2 + z^2 + 2xz - y^2 = y^2 + yz + xy$

i.e., $\quad (x + z)^2 - y^2 = y\,(x + y + z)$

i.e., $\quad x + z - y = y$

i.e., $\quad x + z = 2y$

which is true, since x, y, z are in A.P. Hence $x^2 + xy + y^2$, $z^2 + xz + x^2$, $y^2 + yz + z^2$ are in A.P.

Example 8 If a, b, c, d are in G.P., prove that $a^2 - b^2$, $b^2 - c^2$, $c^2 - d^2$ are also in G.P.

Solution Let r be the common ratio of the given G.P. Then

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r$$

$\Rightarrow \quad b = ar,\ c = br = ar^2,\ d = cr = ar^3$

Now, $\quad a^2 - b^2 = a^2 - a^2r^2 = a^2\,(1 - r^2)$

$$b^2 - c^2 = a^2r^2 - a^2r^4 = a^2r^2\,(1 - r^2)$$

and $$c^2 - d^2 = a^2r^4 - a^2r^6 = a^2r^4\,(1 - r^2)$$

Therefore, $$\frac{b^2 - c^2}{a^2 - b^2} = \frac{c^2 - d^2}{b^2 - c^2} = r^2$$

Hence, $a^2 - b^2$, $b^2 - c^2$, $c^2 - d^2$ are in G.P.

Long Answer Type

Example 9 If the sum of m terms of an A.P. is equal to the sum of either the next n terms or the next p terms, then prove that

$$(m + n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m + p)\left(\frac{1}{m} - \frac{1}{n}\right)$$

Solution Let the A.P. be a, $a + d$, $a + 2d$, ... We are given

$$a_1 + a_2 + ... + a_m = a_{m+1} + a_{m+2} + ... + a_{m+n} \qquad ... (1)$$

Adding $a_1 + a_2 + ... + a_m$ on both sides of (1), we get

$$2\,[a_1 + a_2 + ... + a_m] = a_1 + a_2 + ... + a_m + a_{m+1} + ... + a_{m+n}$$

$$2\,S_m = S_{m+n}$$

Therefore, $$2\frac{m}{2}\{2a + (m-1)d\} = \frac{m+n}{2}\{2a + (m+n-1)d\}$$

Putting $2a + (m - 1)\,d = x$ in the above equation, we get

$$mx = \frac{m+n}{2}\,(x + nd)$$

$$(2m - m - n)\,x = (m + n)\,nd$$

$$\Rightarrow \qquad (m - n)\,x = (m + n)\,nd \qquad ... (2)$$

Similarly, if $a_1 + a_2 + ... + a_m = a_{m+1} + a_{m+2} + ... + a_{m+p}$

Adding $a_1 + a_2 + ... + a_m$ on both sides

we get, $$2\,(a_1 + a_2 + ... + a_m) = a_1 + a_2 + ... + a_{m+1} + ... + a_{m+p}$$

or, $$2\,S_m = S_{m+p}$$

$$\Rightarrow \qquad 2\left[\frac{m}{2}\{2a + (m-1)d\}\right] = \frac{m+p}{2}\,\{2a + (m + p - 1)d\} \text{ which gives}$$

i.e., $$(m - p)\,x = (m + p)pd \qquad ... (3)$$

Dividing (2) by (3), we get

$$\frac{(m-n)x}{(m-p)x} = \frac{(m+n)nd}{(m+p)pd}$$

$\Rightarrow$ $$(m-n)(m+p)p = (m-p)(m+n)n$$

Dividing both sides by mnp, we get

$$(m+p)\left(\frac{1}{n}-\frac{1}{m}\right) = (m+n)\left(\frac{1}{p}-\frac{1}{m}\right)$$

$$= (m+n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$$

Example 10 If $a_1, a_2, ..., a_n$ are in A.P. with common difference d (where $d \neq 0$); then the sum of the series $\sin d$ (cosec a_1 cosec a_2 + cosec a_2 cosec a_3 + ...+ cosec a_{n-1} cosec a_n) is equal to $\cot a_1 - \cot a_n$

Solution We have

$\sin d$ (cosec a_1 cosec a_2 + cosec a_2 cosec a_3 + ...+ cosec a_{n-1} cosec a_n)

$$= \sin d\left[\frac{1}{\sin a_1 \sin a_2} + \frac{1}{\sin a_2 \sin a_3} + ... + \frac{1}{\sin a_{n-1}\sin a_n}\right]$$

$$= \frac{\sin(a_2-a_1)}{\sin a_1 \sin a_2} + \frac{\sin(a_3-a_2)}{\sin a_2 \sin a_3} + ... + \frac{\sin(a_n-a_{n-1})}{\sin a_{n-1}\sin a_n}$$

$$= \frac{\sin a_2 \cos a_1 - \cos a_2 \sin a_1)}{\sin a_1 \sin a_2} + \frac{\sin a_3 \cos a_2 - \cos a_3 \sin a_2)}{\sin a_2 \sin a_3} + ... + \frac{\sin a_n \cos a_{n-1} - \cos a_n \sin a_{n-1})}{\sin a_{n-1}\sin a_n}$$

$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + ... + (\cot a_{n-1} - \cot a_n)$

$= \cot a_1 - \cot a_n$

Example 11

(i) If a, b, c, d are four distinct positive quantities in A.P., then show that $bc > ad$

(ii) If a, b, c, d are four distinct positive quantities in G.P., then show that $a + d > b + c$

Solution

(i) Since a, b, c, d are in A.P., then A.M. > G.M., for the first three terms.

Therefore, $b > \sqrt{ac}$ $\left(\text{Here } \dfrac{a+c}{2} = b\right)$

Squaring, we get $b^2 > ac$ ... (1)

Similarly, for the last three terms

AM > GM

$c > \sqrt{bd}$ $\left(\text{Here } \dfrac{b+d}{2} = c\right)$

$c^2 > bd$ ... (2)

Multiplying (1) and (2), we get

$b^2c^2 > (ac)\ (bd)$

$\Rightarrow$ $bc > ad$

(ii) Since a, b, c, d are in G.P.

again A.M. > G.M. for the first three terms

$\dfrac{a+c}{2} > b$ $\left(\text{since } \sqrt{ac} = b\right)$

$\Rightarrow$ $a + c > 2b$ ... (3)

Similarly, for the last three terms

$\dfrac{b+d}{2} > c$ $\left(\text{since } \sqrt{bd} = c\right)$

$\Rightarrow$ $b + d > 2c$ ... (4)

Adding (3) and (4), we get

$(a + c) + (b + d) > 2b + 2c$

$a + d > b + c$

Eample 12 If a, b, c are three consecutive terms of an A.P. and x, y, z are three consecutive terms of a G.P. Then prove that

$x^{b-c} \cdot y^{c-a} \cdot z^{a-b} = 1$

Solution We have a, b, c as three consecutive terms of A.P. Then

$b - a = c - b = d$ (say)

$c - a = 2d$

$a - b = -d$

Now $x^{b-c} \cdot y^{c-a} \cdot z^{a-b} = x^{-d} \cdot y^{2d} \cdot z^{-d}$

$= x^{-d} (\sqrt{xz})^{2d} \cdot z^{-d}$ (since $y = (\sqrt{xz})$) as x, y, z are G.P.)

$= x^{-d} \cdot x^{d} \cdot z^{d} \cdot z^{-d}$

$= x^{-d+d} \cdot z^{d-d}$

$= x^{o} z^{o} = 1$

Example 13 Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfies $f(x + y) = f(x) \cdot f(y)$ for all natural numbers x, y and further $f(1) = 2$.

Solution Given that

$$f(x+y) = f(x) \cdot f(y) \text{ and } f(1) = 2$$

Therefore,

$$f(2) = f(1+1) = f(1) \cdot f(1) = 2^2$$

$$f(3) = f(1+2) = f(1) \cdot f(2) = 2^3$$

$$f(4) = f(1+3) = f(1) \cdot f(3) = 2^4$$

and so on. Continuing the process, we obtain

$$f(k) = 2^k \text{ and } f(a) = 2^a$$

Hence

$$\sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a) \cdot f(k)$$

$$= f(a) \sum_{k=1}^{n} f(k)$$

$$= 2^a (2^1 + 2^2 + 2^3 + \ldots + 2^n)$$

$$= 2^a \left\{ \frac{2 \cdot (2^n - 1)}{2 - 1} \right\} = 2^{a+1} (2^n - 1) \qquad \ldots (1)$$

But, we are given

$$\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$$

$$2^{a+1}(2^n - 1) = 16(2^n - 1)$$

$\Rightarrow$ $2^{a+1} = 2^4 \Rightarrow a + 1 = 4$

$\Rightarrow$ $a = 3$

Objective Type Questions

Choose the correct answer out of the four given options in Examples 14 to 23 (M.C.Q.).

Example 14 A sequence may be defined as a

(A) relation, whose range $\subseteq$ **N** (natural numbers)

(B) function whose range $\subseteq$ **N**

(C) function whose domain $\subseteq$ **N**

(D) progression having real values

Solution (C) is the correct answer. A sequence is a function $f : \mathbf{N} \to X$ having domain $\subseteq \mathbf{N}$

Example 15 If x, y, z are positive integers then value of expression $(x + y)(y + z)(z + x)$ is

(A) $= 8xyz$ (B) $> 8xyz$ (C) $< 8xyz$ (D) $= 4xyz$

Solution (B) is the correct answer, since

A.M. > G.M., $\frac{x+y}{2} > \sqrt{xy}$, $\frac{y+z}{2} > \sqrt{yz}$ and $\frac{z+x}{2} > \sqrt{zx}$

Multiplying the three inequalities, we get

$$\frac{x+y}{2}.\frac{y+z}{2}.\frac{y+z}{2} > \sqrt{(xy)(yz)(zx)}$$

or, $(x + y)(y + z)(z + x) > 8\,xyz$

Example 16 In a G.P. of positive terms, if any term is equal to the sum of the next two terms. Then the common ratio of the G.P. is

(A) sin 18° (B) 2 cos18° (C) cos 18° (D) 2 sin 18°

Solution (D) is the correct answer, since

$t_n = t_{n+1} + t_{n+2}$

$\Rightarrow \quad ar^{n-1} = ar^n + ar^{n+1}$

$\Rightarrow \quad 1 = r + r^2$

$$r = \frac{-1 \pm \sqrt{5}}{2}, \text{ since } r > 0$$

Therefore, $r = 2\frac{\sqrt{5}-1}{4} = 2 \sin 18°$

Example 17 In an A.P. the pth term is q and the $(p+q)^{th}$ term is 0. Then the qth term is

(A) $-p$ (B p (C) $p+q$ (D) $p-q$

Solution (B) is the correct answer

Let a, d be the first term and common difference respectively.

Therefore, $T_p = a + (p-1)\,d = q$ and ... (1)

$T_{p+q} = a + (p+q-1)\,d = 0$ ... (2)

Subtracting (1), from (2) we get $qd = -q$

Substituting in (1) we get $a = q - (p-1)(-1) = q + p - 1$

Now $T_q = a + (q-1)\,d = q + p - 1 + (q-1)(-1)$

$= q + p - 1 - q + 1 = p$

Example 18 Let S be the sum, P be the product and R be the sum of the reciprocals of 3 terms of a G.P. Then $P^2 R^3 : S^3$ is equal to

(A) 1 : 1 (B) (common ratio)^n : 1

(C) (first term)^2 : (common ratio)^2 (D) none of these

Solution (A) is the correct answer

Let us take a G.P. with three terms $\frac{a}{r}, a, ar$. Then

$$S = \frac{a}{r} + a + ar = \frac{a(r^2+r+1)}{r}$$

$$P = a^3,\ R = \frac{r}{a} + \frac{1}{a} + \frac{1}{ar} = \frac{1}{a}\left(\frac{r^2+r+1}{r}\right)$$

$$\frac{P^2R^3}{S^3} = \frac{a^6 \cdot \frac{1}{a^3}\left(\frac{r^2+r+1}{r}\right)^3}{a^3\left(\frac{r^2+r+1}{r}\right)^3} = 1$$

Therefore, the ratio is 1 : 1

Example 19 The 10th common term between the series

3 + 7 + 11 + ... and 1 + 6 + 11 + ... is

(A) 191 (B) 193 (C) 211 (D) None of these

Solution (A) is the correct answer.

The first common term is 11.

Now the next common term is obtained by adding L.C.M. of the common difference 4 and 5, i.e., 20.

Therefore, 10^{th} common term = T_{10} of the AP whose a = 11 and d = 20

$T_{10} = a + 9d = 11 + 9(20) = 191$

Example 20 In a G.P. of even number of terms, the sum of all terms is 5 times the sum of the odd terms. The common ratio of the G.P. is

(A) $\frac{-4}{5}$ (B) $\frac{1}{5}$ (C) 4 (D) none the these

Solution (C) is the correct answer

Let us consider a G.P. $a, ar, ar^2, \ldots$ with $2n$ terms. We have $\frac{a(r^{2n}-1)}{r-1} = \frac{5a\left((r^2)^n-1\right)}{r^2-1}$

(Since common ratio of odd terms will be r^2 and number of terms will be n)

$$\Rightarrow \frac{a(r^{2n}-1)}{r-1} = 5\frac{a(r^{2n}-1)}{(r^2-1)}$$

$\Rightarrow a(r+1) = 5a$, i.e., $r = 4$

Example 21 The minimum value of the expression $3^x + 3^{1-x}$, $x \in \mathbf{R}$, is

(A) 0 (B) $\frac{1}{3}$ (C) 3 (D) $2\sqrt{3}$

Solution (D) is the correct answer.

We know A.M. ≥ G.M. for positive numbers.

Therefore, $\frac{3^x+3^{1-x}}{2} \geq \sqrt{3^x \cdot 3^{1-x}}$

$$\Rightarrow \frac{3^x+3^{1-x}}{2} \geq \sqrt{3^x \cdot \frac{3}{3^x}}$$

$$\Rightarrow 3^x + 3^{1-x} \geq 2\sqrt{3}$$

9.3 EXERCISE

Short Answer Type

1. The first term of an A.P.is a, and the sum of the first p terms is zero, show that the sum of its next q terms is $\frac{-a(p+q)q}{p-1}$. [**Hint:** Required sum = $S_{p+q} - S_p$]

2. A man saved Rs 66000 in 20 years. In each succeeding year after the first year he saved Rs 200 more than what he saved in the previous year. How much did he save in the first year?

3. A man accepts a position with an initial salary of Rs 5200 per month. It is understood that he will receive an automatic increase of Rs 320 in the very next month and each month thereafter.

   (a) Find his salary for the tenth month

   (b) What is his total earnings during the first year?

4. If the pth and qth terms of a G.P. are q and p respectively, show that its $(p+q)^{th}$ term is $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$.

5. A carpenter was hired to build 192 window frames. The first day he made five frames and each day, thereafter he made two more frames than he made the day before. How many days did it take him to finish the job?

6. We know the sum of the interior angles of a triangle is 180°. Show that the sums of the interior angles of polygons with 3, 4, 5, 6, ... sides form an arithmetic progression. Find the sum of the interior angles for a 21 sided polygon.

7. A side of an equilateral triangle is 20cm long. A second equilateral triangle is inscribed in it by joining the mid points of the sides of the first triangle. The process is continued as shown in the accompanying diagram. Find the perimeter of the sixth inscribed equilateral triangle.

8. In a potato race 20 potatoes are placed in a line at intervals of 4 metres with the first potato 24 metres from the starting point. A contestant is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing back all the potatoes?

9. In a cricket tournament 16 school teams participated. A sum of Rs 8000 is to be awarded among themselves as prize money. If the last placed team is awarded

Rs 275 in prize money and the award increases by the same amount for successive finishing places, how much amount will the first place team receive?

10. If $a_1, a_2, a_3, ..., a_n$ are in A.P., where $a_i > 0$ for all i, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

11. Find the sum of the series

$(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + ...$ to (i) n terms (ii) 10 terms

12. Find the r^{th} term of an A.P. sum of whose first n terms is $2n + 3n^2$.

[**Hint:** $a_n = S_n - S_{n-1}$]

Long Answer Type

13. If A is the arithmetic mean and G_1, G_2 be two geometric means between any two numbers, then prove that

$$2A=\frac{G_1^2}{G_2}+\frac{G_2^2}{G_1}$$

14. If $\theta_1, \theta_2, \theta_3, ..., \theta_n$ are in A.P., whose common difference is d, show that

$$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + ... + \sec\theta_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}.$$

15. If the sum of p terms of an A.P. is q and the sum of q terms is p, show that the sum of $p + q$ terms is $-(p + q)$. Also, find the sum of first $p - q$ terms ($p > q$).

16. If p^{th}, q^{th}, and r^{th} terms of an A.P. and G.P. are both a, b and c respectively, show that

$$a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1$$

Objective Type Questions

Choose the correct answer out of the four given options in each of the Exercises 17 to 26 (M.C.Q.).

17. If the sum of n terms of an A.P. is given by

$S_n = 3n + 2n^2$, then the common difference of the A.P. is

(A) 3 (B) 2 (C) 6 (D) 4

18. The third term of G.P. is 4. The product of its first 5 terms is

(A) 4^3 (B) 4^4 (C) 4^5 (D) None of these

19. If 9 times the 9^{th} term of an A.P. is equal to 13 times the 13^{th} term, then the 22^{nd} term of the A.P. is

(A) 0 (B) 22 (C) 220 (D) 198

20. If x, $2y$, $3z$ are in A.P., where the distinct numbers x, y, z are in G.P. then the common ratio of the G.P. is

(A) 3 (B) $\frac{1}{3}$ (C) 2 (D) $\frac{1}{2}$

21. If in an A.P., $S_n = q\,n^2$ and $S_m = qm^2$, where S_r denotes the sum of r terms of the A.P., then S_q equals

(A) $\frac{q^3}{2}$ (B) mnq (C) q^3 (D) $(m + n)\,q^2$

22. Let S_n denote the sum of the first n terms of an A.P. If $S_{2n} = 3S_n$ then $S_{3n} : S_n$ is equal to

(A) 4 (B) 6 (C) 8 (D) 10

23. The minimum value of $4^x + 4^{1-x}$, $x \in$ R, is

(A) 2 (B) 4 (C) 1 (D) 0

24. Let S_n denote the sum of the cubes of the first n natural numbers and s_n denote the sum of the first n natural numbers. Then $\sum_{r=1}^{n} \frac{S_r}{s_r}$ equals

(A) $\frac{n(n+1)(n+2)}{6}$ (B) $\frac{n(n+1)}{2}$

(C) $\frac{n^2+3n+2}{2}$ (D) None of these

25. If t_n denotes the nth term of the series 2 + 3 + 6 + 11 + 18 + ... then t_{50} is

(A) $49^2 - 1$ (B) 49^2 (C) $50^2 + 1$ (D) $49^2 + 2$

26. The lengths of three unequal edges of a rectangular solid block are in G.P. The volume of the block is 216 cm^3 and the total surface area is 252cm^2. The length of the longest edge is

(A) 12 cm (B) 6 cm (C) 18 cm (D) 3 cm

Fill in the blanks in the Exercises 27 to 29.

27. For a, b, c to be in G.P. the value of $\dfrac{a-b}{b-c}$ is equal to ....

28. The sum of terms equidistant from the beginning and end in an A.P. is equal to ....

29. The third term of a G.P. is 4, the product of the first five terms is ....

State whether statement in Exercises 30 to 34 are True or False.

30. Two sequences cannot be in both A.P. and G.P. together.

31. Every progression is a sequence but the converse, i.e., every sequence is also a progression need not necessarily be true.

32. Any term of an A.P. (except first) is equal to half the sum of terms which are equidistant from it.

33. The sum or difference of two G.P.s, is again a G.P.

34. If the sum of n terms of a sequence is quadratic expression then it always represents an A.P.

Match the questions given under Column I with their appropriate answers given under the Column II.

35.

	Column I		**Column II**
(a)	$4, 1, \frac{1}{}$	(i)	A.P.
(b)	2, 3, 5, 7	(ii)	sequence
(c)	13, 8, 3, –2, –7	(iii)	G.P.

36.

	Column I		**Column II**
(a)	$1^2 + 2^2 + 3^2 + ... + n^2$	(i)	$\left(\dfrac{n(n+1)}{2}\right)^2$
(b)	$1^3 + 2^3 + 3^3 + ... + n^3$	(ii)	$n\ (n + 1)$
(c)	$2 + 4 + 6 + ... + 2n$	(iii)	$\dfrac{n(n+1)(2n+1)}{6}$
(d)	$1 + 2 + 3 + ... + n$	(iv)	$\dfrac{n(n+1)}{2}$

Chapter 10

STRAIGHT LINES

10.1 Overview

10.1.1 *Slope of a line*

If θ is the angle made by a line with positive direction of x-axis in anticlockwise direction, then the value of tan θ is called the **slope of the line** and is denoted by m.

The slope of a line passing through points P(x_1, y_1) and Q (x_2, y_2) is given by

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

10.1.2 ***Angle between two lines*** The angle θ between the two lines having slopes m_1 and m_2 is given by

$$\tan\theta = \pm \frac{(m_1 - m_2)}{1 + m_1 m_2}$$

If we take the acute angle between two lines, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

If the lines are parallel, then $m_1 = m_2$.
If the lines are perpendicular, then $m_1 m_2 = -1$.

10.1.3 ***Collinearity of three points*** If three points P (h, k), Q (x_1, y_1) and R (x_2, y_2) are such that slope of PQ = slope of QR, i.e., $\frac{y_1 - k}{x_1 - h} = \frac{y_2 - y_1}{x_2 - x_1}$

or $(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1)$ then they are said to be collinear.

10.1.4 *Various forms of the equation of a line*

(i) If a line is at a distance a and parallel to x-axis, then the equation of the line is $y = \pm a$.

(ii) If a line is parallel to y-axis at a distance b from y-axis then its equation is $x = \pm b$

(iii) Point-slope form : The equation of a line having slope m and passing through the point (x_0, y_0) is given by $y - y_0 = m(x - x_0)$

(iv) Two-point-form : The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

(v) Slope intercept form : The equation of the line making an intercept c on y-axis and having slope m is given by

$$y = mx + c$$

Note that the value of c will be positive or negative as the intercept is made on the positive or negative side of the y-axis, respectively.

(vi) Intercept form : The equation of the line making intercepts a and b on x- and y-axis respectively is given by $\frac{x}{a} + \frac{y}{b} = 1$.

(vii) Normal form : Suppose a non-vertical line is known to us with following data:

(a) Length of the perpendicular (normal) p from origin to the line.

(b) Angle ω which normal makes with the positive direction of x-axis.

Then the equation of such a line is given by $x \cos \omega + y \sin \omega = p$

10.1.5 *General equation of a line*

Any equation of the form Ax + By + C = 0, where A and B are simultaneously not zero, is called the general equation of a line.

Different forms of Ax + By + C = 0

The general form of the line can be reduced to various forms as given below:

(i) Slope intercept form : If B $\neq$ 0, then Ax + By + C = 0 can be written as

$y = \frac{-A}{B}x + \frac{-C}{B}$ or $y = mx + c$, where $m = \frac{-A}{B}$ and $c = \frac{-C}{B}$

If B = 0, then $x = \frac{-C}{A}$ which is a vertical line whose slope is not defined and x-intercept is $\frac{-C}{A}$.

(ii) Intercept form : If C $\neq$ 0, then Ax + By + C = 0 can be written as $\frac{x}{\frac{-C}{A}} + \frac{y}{\frac{-C}{B}}$ = 1 or $\frac{x}{a}+\frac{y}{b}=1$, where $a=\frac{-C}{A}$ and $b=\frac{-C}{B}$.

If C = 0, then Ax + By + C = 0 can be written as Ax + By = 0 which is a line passing through the origin and therefore has zero intercepts on the axes.

(iii) Normal Form : The normal form of the equation Ax + By + C = 0 is $x \cos\omega + y \sin\omega = p$ where,

$$\cos\omega = \pm\frac{A}{\sqrt{A^2+B^2}}, \sin\omega = \pm\frac{B}{\sqrt{A^2+B^2}} \text{ and } p = \pm\frac{C}{\sqrt{A^2+B^2}}.$$

Note: Proper choice of signs is to be made so that p should be always positive.

10.1.6 ***Distance of a point from a line*** The perpendicular distance (or simply distance) d of a point P (x_1, y_1) from the line Ax + By + C = 0 is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}$$

Distance between two parallel lines

The distance d between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by

$$d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}.$$

10.1.7 ***Locus and Equation of Locus*** The curve described by a point which moves under certain given condition is called its locus. To find the locus of a point P whose coordinates are (h, k), express the condition involving h and k. Eliminate variables if any and finally replace h by x and k by y to get the locus of P.

10.1.8 ***Intersection of two given lines*** Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) intersecting if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(ii) parallel and distinct if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(iii) coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Remarks

(i) The points (x_1, y_1) and (x_2, y_2) are on the same side of the line or on the opposite side of the line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign or of opposite signs respectively.

(ii) The condition that the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c = 0$ are perpendicular is $a_1a_2 + b_1b_2 = 0$.

(iii) The equation of any line through the point of intersection of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $a_1x + b_1y + c_1 + k(ax_2 + by_2 + c_2) = 0$. The value of k is determined from extra condition given in the problem.

10.2 Solved Examples

Short Answer Type

Example 1 Find the equation of a line which passes through the point (2, 3) and makes an angle of 30° with the positive direction of x-axis.

Solution Here the slope of the line is $m = \tan\theta = \tan 30° = \frac{1}{\sqrt{3}}$ and the given point is (2, 3). Therefore, using point slope formula of the equation of a line, we have

$$y - 3 = \frac{1}{\sqrt{3}}(x - 2) \qquad \text{or} \qquad x - \sqrt{3}y + (3\sqrt{3} - 2) = 0.$$

Example 2 Find the equation of the line where length of the perpendicular segment from the origin to the line is 4 and the inclination of the perpendicular segment with the positive direction of x-axis is 30°.

Solution The normal form of the equation of the line is $x \cos\omega + y \sin\omega = p$. Here $p = 4$ and $\omega = 30°$. Therefore, the equation of the line is

$$x \cos 30° + y \sin 30° = 4$$

$$x\frac{\sqrt{3}}{2} + y\frac{1}{2} = 4 \qquad \text{or} \qquad \sqrt{3}x + y = 8$$

Example 3 Prove that every straight line has an equation of the form Ax + By + C = 0, where A, B and C are constants.

Proof Given a straight line, either it cuts the y-axis, or is parallel to or coincident with it. We know that the equation of a line which cuts the y-axis (i.e., it has y-intercept) can be put in the form $y = mx + b$; further, if the line is parallel to or coincident with the y-axis, its equation is of the form $x = x_1$, where $x = 0$ in the case of coincidence. Both of these equations are of the form given in the problem and hence the proof.

Example 4 Find the equation of the straight line passing through (1, 2) and perpendicular to the line $x + y + 7 = 0$.

Solution Let m be the slope of the line whose equation is to be found out which is perpendicular to the line $x + y + 7 = 0$. The slope of the given line $y = (-1)\, x - 7$ is -1. Therefore, using the condition of perpendicularity of lines, we have $m \times (-1) = -1$ or $m = 1$ (Why?)

Hence, the required equation of the line is $y - 1 = (1)\,(x - 2)$ or $y - 1 = x - 2 \Rightarrow x - y - 1 = 0$.

Example 5 Find the distance between the lines $3x + 4y = 9$ and $6x + 8y = 15$.

Solution The equations of lines $3x + 4y = 9$ and $6x + 8y = 15$ may be rewritten as

$$3x + 4y - 9 = 0 \qquad \text{and} \qquad 3x + 4y - \frac{15}{2} = 0$$

Since, the slope of these lines are same and hence they are parallel to each other. Therefore, the distance between them is given by

$$\left|\frac{9 - \frac{15}{2}}{\sqrt{3^2 + 4^2}}\right| = \frac{3}{10}$$

Example 6 Show that the locus of the mid-point of the distance between the axes of the variable line $x \cos\alpha + y \sin\alpha = p$ is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$ where p is a constant.

Solution Changing the given equation of the line into intercept form, we have $\dfrac{x}{\frac{p}{\cos\alpha}} + \dfrac{y}{\frac{p}{\sin\alpha}} = 1$ which gives the coordinates $\left(\frac{p}{\cos\alpha}, 0\right)$ and $\left(0, \frac{p}{\sin\alpha}\right)$, where the line intersects x-axis and y-axis, respectively.

Let (h, k) denote the mid-point of the line segment joining the points $\left(\frac{p}{\cos\alpha}, 0\right)$ and $\left(0, \frac{p}{\sin\alpha},\right)$

Then $h = \frac{p}{2\cos\alpha}$ and $k = \frac{p}{2\sin\alpha}$ (Why?)

This gives $\cos\alpha = \frac{p}{2h}$ and $\sin\alpha = \frac{p}{2k}$

Squaring and adding we get

$$\frac{p^2}{4h^2} + \frac{p^2}{4k^2} = 1 \quad \text{or} \quad \frac{1}{h^2} + \frac{1}{k^2} = \frac{4}{p^2}.$$

Therefore, the required locus is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$.

Example 7 If the line joining two points A(2, 0) and B(3, 1) is rotated about A in anticlock wise direction through an angle of 15°. Find the equation of the line in new position.

Solution The slope of the line AB is $\frac{1-0}{3-2} = 1$ or $\tan 45°$ (Why?) (see Fig.). After rotation of the line through 15°, the slope of the line AC in new position is tan 60° = $\sqrt{3}$

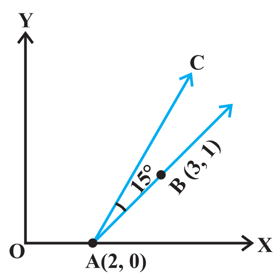


Fig. 10.1

Therefore, the equation of the new line AC is

$$y - 0 = \sqrt{3}(x-2) \qquad \text{or} \quad y - \sqrt{3}x + 2\sqrt{3} = 0$$

Long Answer Type

Example 8 If the slope of a line passing through the point A(3, 2) is $\frac{3}{4}$, then find points on the line which are 5 units away from the point A.

Solution Equation of the line passing through (3, 2) having slope $\frac{3}{4}$ is given by

$$y - 2 = \frac{3}{4}(x-3)$$

or $\quad 4y - 3x + 1 = 0 \qquad (1)$

Let (h, k) be the points on the line such that

$$(h-3)^2 + (k-2)^2 = 25 \qquad (2) \quad \text{(Why?)}$$

Also, we have

$$4k - 3h + 1 = 0 \qquad (3) \quad \text{(Why?)}$$

or $\quad k = \frac{3h-1}{4} \qquad (4)$

Putting the value of k in (2) and on simplifying, we get

$$25h^2 - 150h - 175 = 0 \qquad \text{(How?)}$$

or $\quad h^2 - 6h - 7 = 0$

or $\quad (h+1)(h-7) = 0 \Rightarrow h = -1, h = 7$

Putting these values of k in (4), we get $k = -1$ and $k = 5$. Therefore, the coordinates of the required points are either $(-1, -1)$ or $(7, 5)$.

Example 9 Find the equation to the straight line passing through the point of intersection of the lines $5x - 6y - 1 = 0$ and $3x + 2y + 5 = 0$ and perpendicular to the line $3x - 5y + 11 = 0$.

Solution First we find the point of intersection of lines $5x - 6y - 1 = 0$ and $3x + 2y + 5 = 0$ which is $(-1, -1)$. Also the slope of the line $3x - 5y + 11 = 0$ is $\frac{3}{5}$. Therefore, the slope of the line perpendicular to this line is $\frac{-5}{3}$ (Why?). Hence, the equation of the required line is given by

$$y + 1 = \frac{-5}{3}(x + 1)$$

or $$5x + 3y + 8 = 0$$

Alternatively The equation of any line through the intersection of lines $5x - 6y - 1 = 0$ and $3x + 2y + 5 = 0$ is

$$5x - 6y - 1 + k(3x + 2y + 5) = 0 \qquad (1)$$

or Slope of this line is $\dfrac{-(5+3k)}{-6+2k}$

Also, slope of the line $3x - 5y + 11 = 0$ is $\dfrac{3}{5}$

Now, both are perpendicular

so $$\frac{-(5+3k)}{-6+2k} \times \frac{3}{5} = -1$$

or $$k = 45$$

Therefore, equation of required line in given by

$$5x - 6y - 1 + 45\,(3x + 2y + 5) = 0$$

or $$5x + 3y + 8 = 0$$

Example 10 A ray of light coming from the point (1, 2) is reflected at a point A on the x-axis and then passes through the point (5, 3). Find the coordinates of the point A.

Solution Let the incident ray strike x-axis at the point A whose coordinates be $(x, 0)$. From the figure, the slope of the reflected ray is given by

$$\tan\theta = \frac{3}{5-x} \qquad (1)$$

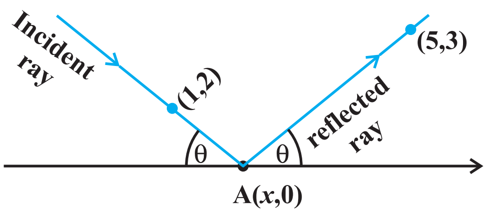


Fig. 10.2

Again, the slope of the incident ray is given by

$$\tan(\pi - \theta) = \frac{-2}{x-1} \qquad \text{(Why?)}$$

or

$$-\tan\theta = \frac{-2}{x-1} \qquad (2)$$

Solving (1) and (2), we get

$$\frac{3}{5-x} = \frac{2}{x-1} \quad \text{or} \quad x = \frac{13}{5}$$

Therefore, the required coordinates of the point A are $\left(\frac{13}{5}, 0\right)$.

Example 11 If one diagonal of a square is along the line $8x - 15y = 0$ and one of its vertex is at (1, 2), then find the equation of sides of the square passing through this vertex.

Solution Let ABCD be the given square and the coordinates of the vertex D be (1, 2). We are required to find the equations of its sides DC and AD.

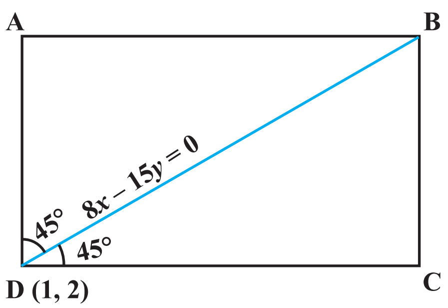


Fig. 10.3

Given that BD is along the line $8x - 15y = 0$, so its slope is $\frac{8}{15}$ (Why?). The angles made by BD with sides AD and DC is 45° (Why?). Let the slope of DC be m. Then

$$\tan 45° = \frac{m - \frac{8}{15}}{1 + \frac{8m}{15}} \qquad \text{(Why?)}$$

or $$15 + 8m = 15m - 8$$

or $$7m = 23, \text{ which gives } m = \frac{23}{7}$$

Therefore, the equation of the side DC is given by

$$y - 2 = \frac{23}{7}(x - 1) \text{ or } 23x - 7y - 9 = 0.$$

Similarly, the equation of another side AD is given by

$$y - 2 = \frac{-7}{23}(x - 1) \text{ or } 7x + 23y - 53 = 0.$$

Objective Type Questions

Each of the Examples 12 to 20 has four possible options out of which only one option is correct. Choose the correct option (M.C.Q.).

Example 12 The inclination of the line $x - y + 3 = 0$ with the positive direction of x-axis is

(A) 45° (B) 135° (C) – 45° (D) –135°

Solution (A) is the correct answer. The equation of the line $x - y + 3 = 0$ can be rewritten as $y = x + 3 \Rightarrow m = \tan\theta = 1$ and hence $\theta = 45°$.

Example 13 The two lines $ax + by = c$ and $a'x + b'y = c'$ are perpendicular if

(A) $aa' + bb' = 0$ (B) $ab' = ba'$

(C) $ab + a'b' = 0$ (D) $ab' + ba' = 0$

Solution (A) is correct answer. Slope of the line $ax + by = c$ is $\frac{-a}{b}$,

and the slope of the line $a'x + b'y = c'$ *is* $\frac{-a'}{b'}$. The lines are perpendicular if

$$\tan\theta = \frac{3}{5-x} \qquad (1)$$

$$\left(\frac{-a}{b}\right)\left(\frac{-a'}{b'}\right) = -1 \text{ or } aa' + bb' = 0 \qquad \text{(Why?)}$$

Example 14 The equation of the line passing through (1, 2) and perpendicular to $x + y + 7 = 0$ is

(A) $y - x + 1 = 0$ (B) $y - x - 1 = 0$

(C) $y - x + 2 = 0$ (D) $y - x - 2 = 0$.

Solution (B) is the correct answer. Let the slope of the line be m. Then, its equation passing through (1, 2) is given by

$$y - 2 = m\,(x - 1) \qquad (1)$$

Again, this line is perpendicular to the given line $x + y + 7 = 0$ whose slope is -1 (Why?)

Therefore, we have $m\,(-1) = -1$

or $m = 1$

Hence, the required equation of the line is obtained by putting the value of m in (1), i.e.,

$$y - 2 = x - 1$$

or $y - x - 1 = 0$

Example 15 The distance of the point P $(1, -3)$ from the line $2y - 3x = 4$ is

(A) 13 (B) $\frac{7}{13}\sqrt{13}$ (C) $\sqrt{13}$ (D) None of these

Solution (A) is the correct answer. The distance of the point P $(1, -3)$ from the line $2y - 3x - 4 = 0$ is the length of perpendicular from the point to the line which is given by

$$\left|\frac{2(-3)-3-4}{\sqrt{13}}\right| = \sqrt{13}$$

Example 16 The coordinates of the foot of the perpendicular from the point (2, 3) on the line $x + y - 11 = 0$ are

(A) $(-6, 5)$ (B) $(5, 6)$ (C) $(-5, 6)$ (D) $(6, 5)$

Solution (B) is the correct choice. Let (h, k) be the coordinates of the foot of the perpendicular from the point (2, 3) on the line $x + y - 11 = 0$. Then, the slope of the perpendicular line is $\frac{k-3}{h-2}$. Again the slope of the given line $x + y - 11 = 0$ is -1 (why?)

Using the condition of perpendicularity of lines, we have

$$\left(\frac{k-3}{h-2}\right)(-1) = -1 \qquad \text{(Why?)}$$

or $k - h = 1 \qquad (1)$

Since (h, k) lies on the given line, we have,

$$h + k - 11 = 0 \text{ or } h + k = 11 \qquad (2)$$

Solving (1) and (2), we get $h = 5$ and $k = 6$. Thus (5, 6) are the required coordinates of the foot of the perpendicular.

Example 17 The intercept cut off by a line from y-axis is twice than that from x-axis, and the line passes through the point (1, 2). The equation of the line is

(A) $2x + y = 4$ (B) $2x + y + 4 = 0$

(C) $2x - y = 4$ (D) $2x - y + 4 = 0$

Solution (A) is the correct choice. Let the line make intercept 'a' on x-axis. Then, it makes intercept '$2a$' on y-axis. Therefore, the equation of the line is given by

$$\frac{x}{a} + \frac{y}{2a} = 1$$

It passes through (1, 2), so, we have

$$\frac{1}{a} + \frac{2}{2a} = 1 \text{ or } a = 2$$

Therefore, the required equation of the line is given by

$$\frac{x}{2} + \frac{y}{4} = 1 \quad \text{or} \quad 2x + y = 4$$

Example 18 A line passes through P (1, 2) such that its intercept between the axes is bisected at P. The equation of the line is

(A) $x + 2y = 5$ (B) $x - y + 1 = 0$

(C) $x + y - 3 = 0$ (D) $2x + y - 4 = 0$

Solution The correct choice is (D). We know that the equation of a line making intercepts a and b with x-axis and y-axis, respectively, is given by

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Here we have $$1 = \frac{a+0}{2} \text{ and } 2 = \frac{0+b}{2}, \qquad \text{(Why?)}$$

which give $a = 2$ and $b = 4$. Therefore, the required equation of the line is given by

$$\frac{x}{2}+\frac{y}{4}=1 \quad \text{or} \quad 2x + y - 4 = 0$$

Example 19 The reflection of the point (4, – 13) about the line $5x + y + 6 = 0$ is

(A) (– 1, – 14) (B) (3, 4) (C) (0, 0) (D) (1, 2)

Solution The correct choice is (A). Let (h, k) be the point of reflection of the given point (4, – 13) about the line $5x + y + 6 = 0$. The mid-point of the line segment joining points (h, k) and (4, – 13) is given by

$$\left(\frac{h+4}{2}, \frac{k-13}{2}\right) \qquad \text{(Why?)}$$

This point lies on the given line, so we have

$$5\left(\frac{h+4}{2}\right)+\frac{k-13}{2}+6=0$$

or $$5h + k + 19 = 0 \qquad (1)$$

Again the slope of the line joining points (h, k) and (4, –13) is given by $\frac{k+13}{h-4}$. This line is perpendicular to the given line and hence $(-5)\left(\frac{k+3}{h-4}\right)=-1$ (Why?)

This gives $$5k + 65 = h - 4$$

or $$h - 5k - 69 = 0 \qquad (2)$$

On solving (1) and (2), we get $h = -1$ and $k = -14$. Thus the point (–1, – 14) is the reflection of the given point.

Example 20 A point moves such that its distance from the point (4, 0) is half that of its distance from the line $x = 16$. The locus of the point is

(A) $3x^2 + 4y^2 = 192$ (B) $4x^2 + 3y^2 = 192$

(C) $x^2 + y^2 = 192$ (D) None of these

Solution The correct choice is (A). Let (h, k) be the coordinates of the moving point. Then, we have

$$\sqrt{(h-4)^2+k^2}=\frac{1}{2}\left(\frac{h-16}{\sqrt{1^2+0}}\right) \qquad \text{(Why?)}$$

$$\Rightarrow \qquad (h - 4)^2 + k^2 = \frac{1}{4}(h - 16)^2$$

$$4(h^2 - 8h + 16 + k^2) = h^2 - 32h + 256$$

or $$3h^2 + 4k^2 = 192$$

Hence, the required locus is given by $3x^2 + 4y^2 = 192$

10.3 EXERCISE

Short Answer Type Questions

1. Find the equation of the straight line which passes through the point (1, – 2) and cuts off equal intercepts from axes.
2. Find the equation of the line passing through the point (5, 2) and perpendicular to the line joining the points (2, 3) and (3, – 1).
3. Find the angle between the lines $y = (2 - \sqrt{3})(x + 5)$ and $y = (2 + \sqrt{3})(x - 7)$.
4. Find the equation of the lines which passes through the point (3, 4) and cuts off intercepts from the coordinate axes such that their sum is 14.
5. Find the points on the line $x + y = 4$ which lie at a unit distance from the line $4x + 3y = 10$.
6. Show that the tangent of an angle between the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{a} - \frac{y}{b} = 1$ is $\frac{2ab}{a^2 - b^2}$.
7. Find the equation of lines passing through (1, 2) and making angle 30° with y-axis.
8. Find the equation of the line passing through the point of intersection of $2x + y = 5$ and $x + 3y + 8 = 0$ and parallel to the line $3x + 4y = 7$.
9. For what values of a and b the intercepts cut off on the coordinate axes by the line $ax + by + 8 = 0$ are equal in length but opposite in signs to those cut off by the line $2x - 3y + 6 = 0$ on the axes.
10. If the intercept of a line between the coordinate axes is divided by the point (–5, 4) in the ratio 1 : 2, then find the equation of the line.
11. Find the equation of a straight line on which length of perpendicular from the origin is four units and the line makes an angle of 120° with the positive direction of x-axis.

[**Hint**: Use normal form, here ω =30°.]

12. Find the equation of one of the sides of an isosceles right angled triangle whose hypotenuse is given by $3x + 4y = 4$ and the opposite vertex of the hypotenuse is (2, 2).

Long Answer Type

13. If the equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is (2, – 1), then find the length of the side of the triangle.

[**Hint**: Find length of perpendicular (p) from (2, – 1) to the line and use p = l sin 60°, where l is the length of side of the triangle].

14. A variable line passes through a fixed point P. The algebraic sum of the perpendiculars drawn from the points (2, 0), (0, 2) and (1, 1) on the line is zero. Find the coordinates of the point P.

[**Hint:** Let the slope of the line be m. Then the equation of the line passing through the fixed point P (x_1, y_1) is $y - y_1 = m(x - x_1)$. Taking the algebraic sum of perpendicular distances equal to zero, we get $y - 1 = m(x - 1)$. Thus (x_1, y_1) is (1, 1).]

15. In what direction should a line be drawn through the point (1, 2) so that its point of intersection with the line $x + y = 4$ is at a distance $\frac{\sqrt{6}}{3}$ from the given point.

16. A straight line moves so that the sum of the reciprocals of its intercepts made on axes is constant. Show that the line passes through a fixed point.

[**Hint:** $\frac{x}{a}+\frac{y}{b}=1$ where $\frac{1}{a}+\frac{1}{b}=$ constant = $\frac{1}{k}$ (say). This implies that $\frac{k}{a}+\frac{k}{b}=1 \Rightarrow$ line passes through the fixed point (k, k).]

17. Find the equation of the line which passes through the point (– 4, 3) and the portion of the line intercepted between the axes is divided internally in the ratio 5 : 3 by this point.

18. Find the equations of the lines through the point of intersection of the lines $x - y + 1 = 0$ and $2x - 3y + 5 = 0$ and whose distance from the point (3, 2) is $\frac{7}{5}$.

19. If the sum of the distances of a moving point in a plane from the axes is 1, then find the locus of the point. [**Hint**: Given that $|x| + |y| = 1$, which gives four sides of a square.]

20. P_1, P_2 are points on either of the two lines $y - \sqrt{3}\,|x| = 2$ at a distance of 5 units from their point of intersection. Find the coordinates of the foot of perpendiculars drawn from P_1, P_2 on the bisector of the angle between the given lines.

[**Hint**: Lines are $y = \sqrt{3}x + 2$ and $y = -\sqrt{3}x + 2$ according as $x \geq 0$ or $x < 0$. y-axis is the bisector of the angles between the lines. P_1, P_2 are the points on these lines at a distance of 5 units from the point of intersection of these lines which have a point on y-axis as common foot of perpendiculars from these points. The y-coordinate of the foot of the perpendicular is given by $2 + 5\cos 30°$.]

21. If p is the length of perpendicular from the origin on the line $\frac{x}{a} + \frac{y}{b} = 1$ and a^2, p^2, b^2 are in A.P, then show that $a^4 + b^4 = 0$.

Objective Type Questions

Choose the correct answer from the given four options in Exercises 22 to 41

22. A line cutting off intercept – 3 from the y-axis and the tengent at angle to the x-axis is $\frac{3}{5}$, its equation is

(A) $5y - 3x + 15 = 0$ (B) $3y - 5x + 15 = 0$

(C) $5y - 3x - 15 = 0$ (D) None of these

23. Slope of a line which cuts off intercepts of equal lengths on the axes is

(A) -1 (B) -0

(C) 2 (D) $\sqrt{3}$

24. The equation of the straight line passing through the point (3, 2) and perpendicular to the line $y = x$ is

(A) $x - y = 5$ (B) $x + y = 5$

(C) $x + y = 1$ (D) $x - y = 1$

25. The equation of the line passing through the point (1, 2) and perpendicular to the line $x + y + 1 = 0$ is

(A) $y - x + 1 = 0$ (B) $y - x - 1 = 0$

(C) $y - x + 2 = 0$ (D) $y - x - 2 = 0$

26. The tangent of angle between the lines whose intercepts on the axes are a, $-b$ and b, $-a$, respectively, is

(A) $\frac{a^2-b^2}{ab}$ (B) $\frac{b^2-a^2}{2}$

(C) $\frac{b^2-a^2}{2ab}$ (D) None of these

27. If the line $\frac{x}{a}+\frac{y}{b}=1$ passes through the points (2, –3) and (4, –5), then (a, b) is

(A) (1, 1) (B) (– 1, 1) (C) (1, – 1) (D) (– 1, –1)

28. The distance of the point of intersection of the lines $2x - 3y + 5 = 0$ and $3x + 4y = 0$ from the line $5x - 2y = 0$ is

(A) $\frac{130}{17\sqrt{29}}$ (B) $\frac{13}{7\sqrt{29}}$ (C) $\frac{130}{7}$ (D) None of these

29. The equations of the lines which pass through the point (3, –2) and are inclined at 60° to the line $\sqrt{3}\, x + y = 1$ is

(A) $y + 2 = 0, \ \sqrt{3}x - y - 2 - 3\sqrt{3} = 0$

(B) $x - 2 = 0, \ \sqrt{3}\, x - y + 2 + 3\sqrt{3} = 0$

(C) $\sqrt{3}\, x - y - 2 - 3\sqrt{3} = 0$

(D) None of these

30. The equations of the lines passing through the point (1, 0) and at a distance $\frac{\sqrt{3}}{2}$ from the origin, are

(A) $\sqrt{3}x + y - \sqrt{3} = 0, \ \sqrt{3}x - y - \sqrt{3} = 0$

(B) $\sqrt{3}x + y + \sqrt{3} = 0, \ \sqrt{3}x - y + \sqrt{3} = 0$

(C) $x + \sqrt{3}\, y - \sqrt{3} = 0, \ x - \sqrt{3}\, y - \sqrt{3} = 0$

(D) None of these.

31. The distance between the lines $y = mx + c_1$ and $y = mx + c_2$ is

(A) $\frac{c_1 - c_2}{\sqrt{m^2+1}}$ (B) $\frac{|c_1 - c_2|}{\sqrt{1+m^2}}$ (C) $\frac{c_2 - c_1}{\sqrt{1+m^2}}$ (D) 0

32. The coordinates of the foot of perpendiculars from the point (2, 3) on the line $y = 3x + 4$ is given by

(A) $\left(\frac{37}{10}, \frac{-1}{10}\right)$ (B) $\left(\frac{-1}{10}, \frac{37}{10}\right)$ (C) $\left(\frac{10}{37}, -10\right)$ (D) $\left(\frac{2}{3}, -\frac{1}{3}\right)$

33. If the coordinates of the middle point of the portion of a line intercepted between the coordinate axes is (3, 2), then the equation of the line will be

(A) $2x + 3y = 12$ (B) $3x + 2y = 12$ (C) $4x - 3y = 6$ (D) $5x - 2y = 10$

34. Equation of the line passing through (1, 2) and parallel to the line $y = 3x - 1$ is

(A) $y + 2 = x + 1$ (B) $y + 2 = 3(x + 1)$

(C) $y - 2 = 3(x - 1)$ (D) $y - 2 = x - 1$

35. Equations of diagonals of the square formed by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$ are

(A) $y = x$, $y + x = 1$ (B) $y = x$, $x + y = 2$

(C) $2y = x$, $y + x = \frac{1}{3}$ (D) $y = 2x$, $y + 2x = 1$

36. For specifying a straight line, how many geometrical parameters should be known?

(A) 1 (B) 2 (C) 4 (D) 3

37. The point (4, 1) undergoes the following two successive transformations :

(i) Reflection about the line $y = x$

(ii) Translation through a distance 2 units along the positive x-axis

Then the final coordinates of the point are

(A) (4, 3) (B) (3, 4) (C) (1, 4) (D) $\left(\frac{7}{2}, \frac{7}{2}\right)$

38. A point equidistant from the lines $4x + 3y + 10 = 0$, $5x - 12y + 26 = 0$ and $7x + 24y - 50 = 0$ is

(A) (1, –1) (B) (1, 1) (C) (0, 0) (D) (0, 1)

39. A line passes through (2, 2) and is perpendicular to the line $3x + y = 3$. Its y-intercept is

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) 1 (D) $\frac{4}{3}$

40. The ratio in which the line $3x + 4y + 2 = 0$ divides the distance between the lines $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ is

(A) 1 : 2 (B) 3 : 7 (C) 2 : 3 (D) 2 : 5

41. One vertex of the equilateral triangle with centroid at the origin and one side as $x + y - 2 = 0$ is

(A) (–1, –1) (B) (2, 2) (C) (–2, –2) (D) (2, –2)

[**Hint**: Let ABC be the equilateral triangle with vertex A (h, k) and let D (α, β) be the point on BC. Then $\frac{2\alpha + h}{3} = 0 = \frac{2\beta + k}{3}$. Also $\alpha + \beta - 2 = 0$ and $\left(\frac{k-0}{h-0}\right) \times (-1) = -1$].

Fill in the blank in Exercises 42 to 47.

42. If a, b, c are in A.P., then the straight lines $ax + by + c = 0$ will always pass through ____.

43. The line which cuts off equal intercept from the axes and pass through the point (1, –2) is ____.

44. Equations of the lines through the point (3, 2) and making an angle of 45° with the line $x - 2y = 3$ are ____.

45. The points (3, 4) and (2, – 6) are situated on the ____ of the line $3x - 4y - 8 = 0$.

46. A point moves so that square of its distance from the point (3, –2) is numerically equal to its distance from the line $5x - 12y = 3$. The equation of its locus is ____.

47. Locus of the mid-points of the portion of the line $x \sin\theta + y \cos\theta = p$ intercepted between the axes is ____.

State whether the statements in Exercises 48 to 56 are true or false. Justify.

48. If the vertices of a triangle have integral coordinates, then the triangle can not be equilateral.

49. The points A (– 2, 1), B (0, 5), C (– 1, 2) are collinear.

50. Equation of the line passing through the point $(a \cos^3\theta, a \sin^3\theta)$ and perpendicular to the line $x \sec\theta + y \operatorname{cosec}\theta = a$ is $x \cos\theta - y \sin\theta = a \sin 2\theta$.

51. The straight line $5x + 4y = 0$ passes through the point of intersection of the straight lines $x + 2y - 10 = 0$ and $2x + y + 5 = 0$.

52. The vertex of an equilateral triangle is (2, 3) and the equation of the opposite side is $x + y = 2$. Then the other two sides are $y - 3 = (2 \pm \sqrt{3})(x - 2)$.

53. The equation of the line joining the point (3, 5) to the point of intersection of the lines $4x + y - 1 = 0$ and $7x - 3y - 35 = 0$ is equidistant from the points (0, 0) and (8, 34).

54. The line $\frac{x}{a}+\frac{y}{b}=1$ moves in such a way that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$, where c is a constant. The locus of the foot of the perpendicular from the origin on the given line is $x^2 + y^2 = c^2$.

55. The lines $ax + 2y + 1 = 0$, $bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent if a, b, c are in G.P.

56. Line joining the points (3, – 4) and (– 2, 6) is perpendicular to the line joining the points (–3, 6) and (9, –18).

Match the questions given under Column C_1 with their appropriate answers given under the Column C_2 in Exercises 57 to 59.

57.

	Column C_1		Column C_2
(a)	The coordinates of the points P and Q on the line $x + 5y = 13$ which are at a distance of 2 units from the line $12x - 5y + 26 = 0$ are	(i)	(3, 1), (–7, 11)
(b)	The coordinates of the point on the line $x + y = 4$, which are at a unit distance from the line $4x + 3y - 10 = 0$ are	(ii)	$\left(-\frac{1}{3},\frac{11}{3}\right),\left(\frac{4}{3},\frac{7}{3}\right)$
(c)	The coordinates of the point on the line joining A (–2, 5) and B (3, 1) such that AP = PQ = QB are	(iii)	$\left(1,\frac{12}{5}\right),\left(-3,\frac{16}{5}\right)$

58. The value of the λ, if the lines $(2x + 3y + 4) + \lambda\,(6x - y + 12) = 0$ are

	Column C_1		Column C_2
(a)	parallel to y-axis is	(i)	$\lambda=-\frac{3}{4}$

(b)	perpendicular to $7x + y - 4 = 0$ is	(ii)	$\lambda = -\frac{1}{3}$
(c)	passes through (1, 2) is	(iii)	$\lambda = -\frac{17}{41}$
(d)	parallel to x axis is	(iv)	$\lambda = 3$

59. The equation of the line through the intersection of the lines $2x - 3y = 0$ and $4x - 5y = 2$ and

	Column C_1		**Column C_2**
(a)	through the point (2, 1) is	(i)	$2x - y = 4$
(b)	perpendicular to the line $x + 2y + 1 = 0$ is	(ii)	$x + y - 5 = 0$
(c)	parallel to the line $3x - 4y + 5 = 0$ is	(iii)	$x - y - 1 = 0$
(d)	equally inclined to the axes is	(iv)	$3x - 4y - 1 = 0$

Chapter 11

CONIC SECTIONS

11.1 Overview

11.1.1 *Sections of a cone* Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and inclined to it at an angle α (Fig. 11.1).

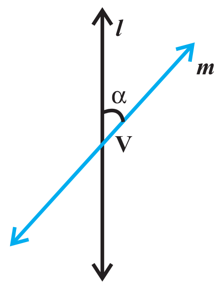


Fig. 11.1

Suppose we rotate the line m around the line l in such a way that the angle α remains constant. Then the surface generated is a double-napped right circular hollow cone herein after referred as cone and extending indefinitely in both directions (Fig. 11.2).

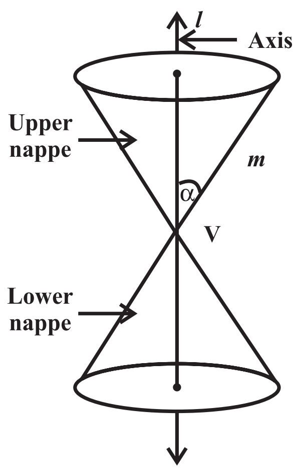


Fig. 11.2

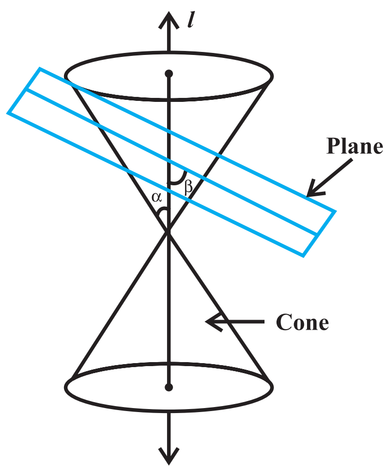


Fig. 11.3

The point V is called the *vertex*; the line l is the *axis* of the cone. The rotating line m is called a *generator* of the cone. The *vertex* separates the cone into two parts called *nappes*.

If we take the intersection of a plane with a cone, the section so obtained is called a *conic section*. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane.

We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and the angle made by it with the vertical axis of the cone. Let β be the angle made by the intersecting plane with the vertical axis of the cone (Fig.11.3).

The intersection of the plane with the cone can take place either at the vertex of the cone or at any other part of the nappe either below or above the vertex.

When the plane cuts the nappe (other than the vertex) of the cone, we have the following situations:

(a) When $\beta = 90^o$, the section is a *circle*.

(b) When $\alpha < \beta < 90^o$, the section is an *ellipse*.

(c) When $\beta = \alpha$; the section is a *parabola*.

(In each of the above three situations, the plane cuts entirely across one nappe of the cone).

(d) When $0 \leq \beta < \alpha$; the plane cuts through both the nappes and the curves of intersection is a *hyperbola*.

Indeed these curves are important tools for present day exploration of outer space and also for research into the behaviour of atomic particles.

We take conic sections as plane curves. For this purpose, it is convenient to use equivalent definition that refer only to the plane in which the curve lies, and refer to special points and lines in this plane called foci and directrices. According to this approach, parabola, ellipse and hyperbola are defined in terms of a fixed point (called focus) and fixed line (called directrix) in the plane.

If S is the focus and l is the directrix, then the set of all points in the plane whose distance from S bears a constant ratio e called eccentricity to their distance from l is a conic section.

As special case of ellipse, we obtain circle for which $e = 0$ and hence we study it differently.

11.1.2 ***Circle*** A circle is the set of all points in a plane which are at a fixed distance from a fixed point in the plane. The fixed point is called the centre of the circle and the distance from centre to any point on the circle is called the radius of the circle.

The equation of a circle with radius r having centre (h, k) is given by $(x - h)^2 + (y - k)^2 = r^2$

The general equation of the circle is given by $x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f and c are constants.

(a) The centre of this circle is $(-g, -f)$

(b) The radius of the circle is $\sqrt{g^2 + f^2 - c}$

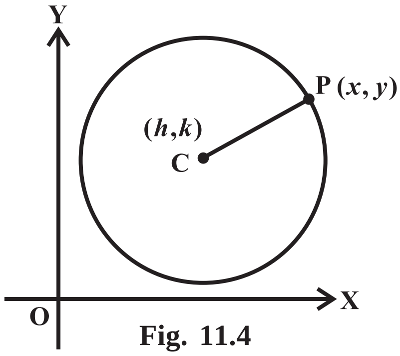


Fig. 11.4

The general equation of the circle passing through the origin is given by $x^2 + y^2 + 2gx + 2fy = 0$.

General equation of second degree i.e., $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a circle if (i) the coefficient of x^2 equals the coefficient of y^2, i.e., $a = b \neq 0$ and (ii) the coefficient of xy is zero, i.e., $h = 0$.

The parametric equations of the circle $x^2 + y^2 = r^2$ are given by $x = r\cos\theta, y = r\sin\theta$ where θ is the parameter and the parametric equations of the circle $(x - h)^2 + (y - k)^2 = r^2$ are given by

$$x - h = r\cos\theta,\ y - k = r\sin\theta$$

or
$$x = h + r\cos\theta,\ y = k + r\sin\theta.$$

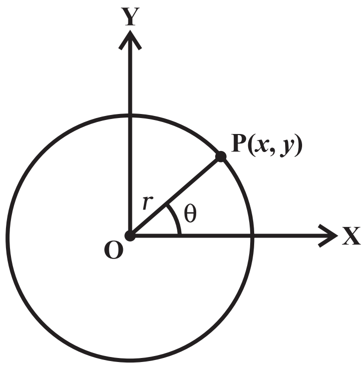


Fig. 11.5

Note: The general equation of the circle involves three constants which implies that at least three conditions are required to determine a circle uniquely.

11.1.3 *Parabola*

A parabola is the set of points P whose distances from a fixed point F in the plane are equal to their distances from a fixed line l in the plane. The fixed point F is called focus and the fixed line l the **directrix** of the parabola.

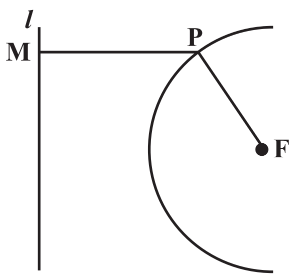


Fig. 11.6

Standard equations of parabola

The four possible forms of parabola are shown below in Fig. 11.7 (a) to (d)

The latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola (Fig. 11.7).

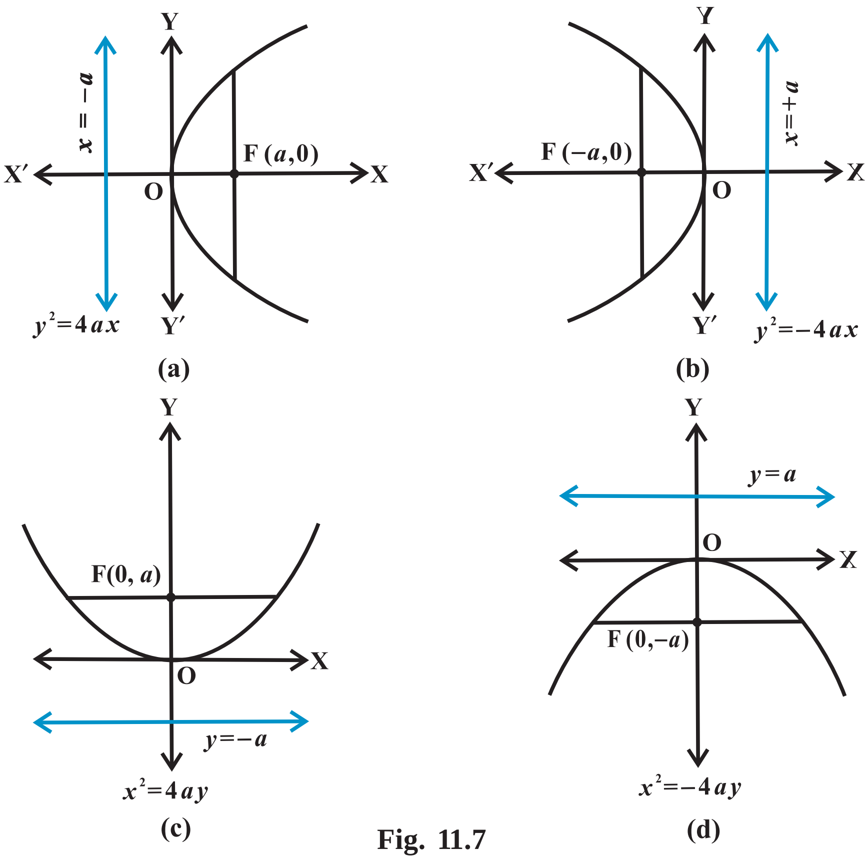


Fig. 11.7

Main facts about the parabola

Forms of Parabolas	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directix	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Equations of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$

Focal distance of a point

Let the equation of the parabola be $y^2 = 4ax$ and P(x, y) be a point on it. Then the distance of P from the focus (a, 0) is called the focal distance of the point, i.e.,

$$\begin{aligned} \text{FP} &= \sqrt{(x-a)^2 + y^2} \\ &= \sqrt{(x-a)^2 + 4ax} \\ &= \sqrt{(x+a)^2} \\ &= |x + a| \end{aligned}$$

11.1.4 *Ellipse* An ellipse is the set of points in a plane, the sum of whose distances from two fixed points is constant. Alternatively, an ellipse is the set of all points in the plane whose distances from a fixed point in the plane bears a constant ratio, less than, to their distance from a fixed line in the plane. The fixed point is called focus, the fixed line a directrix and the constant ratio (e) the centricity of the ellipse.

We have two standard forms of the ellipse, i.e.,

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and (ii) $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$,

In both cases $a > b$ and $b^2 = a^2(1 - e^2)$, $e < 1$.

In (i) major axis is along x-axis and minor along y-axis and in (ii) major axis is along y-axis and minor along x-axis as shown in Fig. 11.8 (a) and (b) respectively.

Main facts about the Ellipse

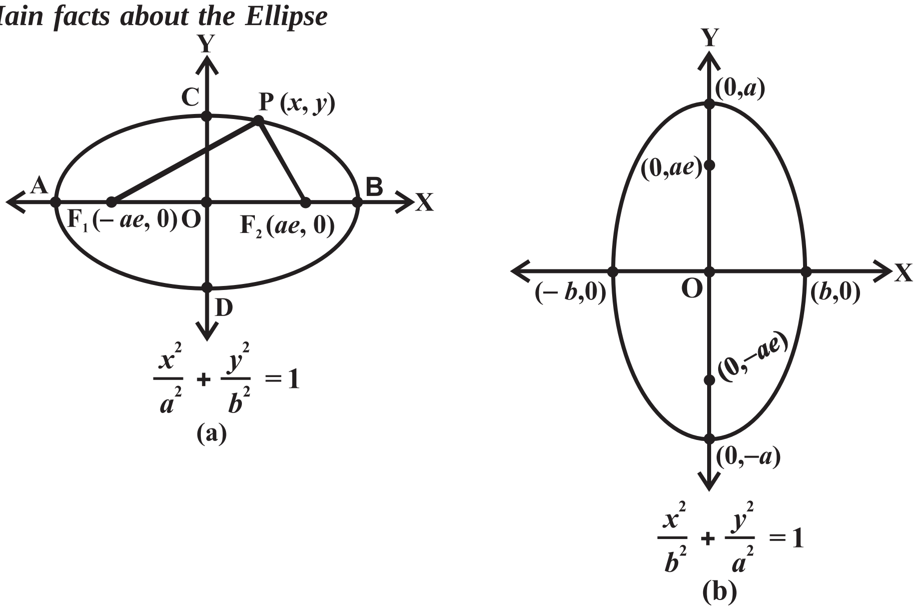


Fig. 11.8

Forms of the ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
	$a > b$	$a > b$
Equation of major axis	$y = 0$	$x = 0$
Length of major axis	$2a$	$2a$
Equation of Minor axis	$x = 0$	$y = 0$
Length of Minor axis	$2b$	$2b$
Directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{a}{e}$
Equation of latus rectum	$x = \pm ae$	$y = \pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Centre	(0, 0)	(0, 0)

Focal Distance

The focal distance of a point (x, y) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$a - e\,|\,x\,|$ from the nearer focus

$a + e\,|\,x\,|$ from the farther focus

Sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis.

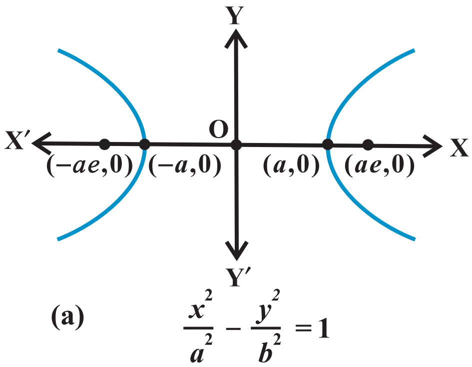
11.1.5 *Hyperbola* A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points is constant. Alternatively, a hyperbola is the set of all points in a plane whose distances from a fixed point in the plane bears a constant ratio, greater than 1, to their distances from a fixed line in the plane. The fixed point is called a focus, the fixed line a directrix and the constant ratio denoted by e, the ecentricity of the hyperbola.

We have two standard forms of the hyperbola, i.e.,

(i) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and (ii) $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

Here $b^2 = a^2 (e^2 - 1)$, $e > 1$.

In (i) transverse axis is along x-axis and conjugate axis along y-axis where as in (ii) transverse axis is along y-axis and conjugate axis along x-axis.



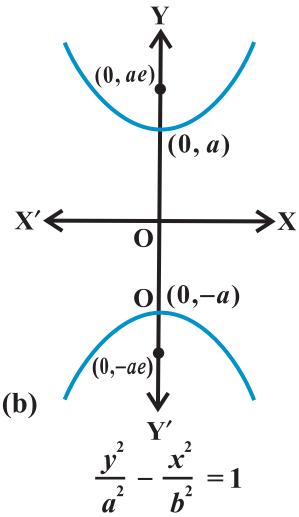


Fig. 11.9

Main facts about the Hyperbola

Forms of the hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugate axis	$x = 0$	$y = 0$
Length of transverse axis	$2a$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm ae)$
Equation of latus rectum	$x = \pm ae$	$y = \pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Centre	$(0, 0)$	$(0, 0)$

Focal distance

The focal distance of any point (x, y) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$e \mid x \mid - a$ from the nearer focus

$e \mid x \mid + a$ from the farther focus

Differences of the focal distances of any point on a hyperbola is constant and equal to the length of the transverse axis.

Parametric equation of conics

	Conics	**Parametric equations**
(i)	Parabola : $y^2 = 4ax$	$x = at^2,\ y = 2at;\ -\infty < t < \infty$
(ii)	Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos\theta,\ y = b\sin\theta;\ 0 \le \theta \le 2\pi$
(iii)	Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec\theta,\ y = b\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2};\ \frac{\pi}{2} < \theta < \frac{3\pi}{2}$

11.2 Solved Examples

Short Answer Type

Example 1 Find the centre and radius of the circle $x^2 + y^2 - 2x + 4y = 8$

Solution we write the given equation in the form $(x^2 - 2x) + (y^2 + 4y) = 8$

Now, completing the squares, we get

$(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 1 + 4$

$(x - 1)^2 + (y + 2)^2 = 13$

Comparing it with the standard form of the equation of the circle, we see that the centre of the circle is $(1, -2)$ and radius is $\sqrt{13}$.

Example 2 If the equation of the parabola is $x^2 = -8y$, find coordinates of the focus, the equation of the directrix and length of latus rectum.

Solution The given equation is of the form $x^2 = -4ay$ where a is positive.

Therefore, the focus is on y-axis in the negative direction and parabola opens downwards.

Comparing the given equation with standard form, we get $a = 2$.

Therefore, the coordinates of the focus are (0, –2) and the the equation of directrix is $y = 2$ and the length of the latus rectum is $4a$, i.e., 8.

Example 3 Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the major and minor axes, eccentricity, foci and vertices.

Solution We put the equation in standard form by dividing by 225 and get

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

This shows that $a = 5$ and $b = 3$. Hence $9 = 25(1 - e^2)$, so $e = \frac{4}{5}$. Since the denominator of x^2 is larger, the major axis is along x-axis, minor axis along y-axis, foci are (4, 0) and (– 4, 0) and vertices are (5, 0) and (–5, 0).

Example 4 Find the equation of the ellipse with foci at (± 5, 0) and $x = \frac{36}{5}$ as one of the directrices.

Solution We have $ae = 5$, $\frac{a}{e} = \frac{36}{5}$ which give $a^2 = 36$ or $a = 6$. Therefore, $e = \frac{5}{6}$.

Now $b = a\sqrt{1-e^2} = 6\sqrt{1-\frac{25}{36}} = \sqrt{11}$. Thus, the equation of the ellipse is $\frac{x^2}{36} + \frac{y^2}{11} = 1$.

Example 5 For the hyperbola $9x^2 - 16y^2 = 144$, find the vertices, foci and eccentricity.

Solution The equation of the hyperbola can be written as $\frac{x^2}{16} - \frac{y^2}{9} = 1$, so $a = 4$, $b = 3$ and $9 = 16(e^2 - 1)$, so that $e^2 = \frac{9}{16} + 1 = \frac{25}{16}$, which gives $e = \frac{5}{4}$. Vertices are $(\pm a, 0) =$ (± 4, 0) and foci are $(\pm ae, 0)$ = (± 5, 0).

Example 6 Find the equation of the hyperbola with vertices at (0, ± 6) and $e = \frac{5}{3}$. Find its foci.

Solution Since the vertices are on the y-axes (with origin at the mid-point), the equation is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

As vertices are $(0, \pm 6)$, $a = 6$, $b^2 = a^2 (e^2 - 1) = 36 \left(\frac{25}{9} - 1\right) = 64$, so the required equation of the hyperbola is $\frac{y^2}{36} - \frac{x^2}{64} = 1$ and the foci are $(0, \pm ae) = (0, \pm 10)$.

Long Answer Type

Example 7 Find the equation of the circle which passes through the points (20, 3), (19, 8) and (2, –9). Find its centre and radius.

Solution By substitution of coordinates in the general equation of the circle given by $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$$40g + 6f + c = -409$$
$$38g + 16f + c = -425$$
$$4g - 18f + c = -85$$

From these three equations, we get

$g = -7, f = -3$ and $c = -111$

Hence, the equation of the circle is

$$x^2 + y^2 - 14x - 6y - 111 = 0$$

or $$(x - 7)^2 + (y - 3)^2 = 13^2$$

Therefore, the centre of the circle is (7, 3) and radius is 13.

Example 8 An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ whose one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Solution As shown in the figure APQ denotes the equilateral triangle with its equal sides of length l (say).

Here $AP = l$ so $AR = l \cos 30°$

$$= l \frac{\sqrt{3}}{2}$$

Also, $PR = l \sin 30° = \frac{l}{2}$.

Thus $\left(\frac{l\sqrt{3}}{2}, \frac{l}{2}\right)$ are the coordinates of the point P lying on the parabola $y^2 = 4ax$.

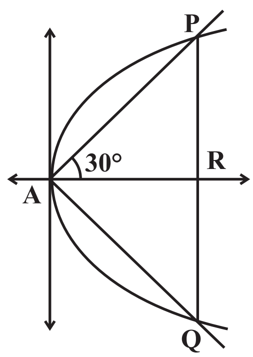


Fig. 11.10

Therefore, $\frac{l^2}{4} = 4a\left(\frac{l\sqrt{3}}{2}\right) \Rightarrow l = 8a\sqrt{3}$.

Thus, $8a\sqrt{3}$ is the required length of the side of the equilateral triangle inscribed in the parabola $y^2 = 4ax$.

Example 9 Find the equation of the ellipse which passes through the point (–3, 1) and has eccentricity $\frac{\sqrt{2}}{5}$, with x-axis as its major axis and centre at the origin.

Solution Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of the ellipse passing through the point (–3, 1).

Therefore, we have $\frac{9}{a^2} + \frac{1}{b^2} = 1$.

or $9b^2 + a^2 = a^2 b^2$

or $9a^2 (a^2 - e^2) + a^2 = a^2 a^2 (1 - e^2)$ (Using $b^2 = a^2 (1 - e^2)$)

or $a^2 = \frac{32}{3}$

Again $b^2 = a^2 (1 - e^2) = \frac{32}{3}\left(1 - \frac{2}{5}\right) = \frac{32}{5}$

Hence, the required equation of the ellipse is

$$\frac{x^2}{\frac{32}{3}} + \frac{y^2}{\frac{32}{5}} = 1$$

or $3x^2 + 5y^2 = 32$.

Example 10 Find the equation of the hyperbola whose vertices are (± 6, 0) and one of the directrices is $x = 4$.

Solution As the vertices are on the x-axis and their middle point is the origin, the equation is of the type $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Here $b^2 = a^2 (e^2 - 1)$, vertices are $(\pm a, 0)$ and directrices are given by $x = \pm \frac{a}{e}$.

Thus $a = 6$, $\frac{a}{e} = 4$ and so $e = \frac{3}{2}$ which gives $b^2 = 36\left(\frac{9}{4} - 1\right) = 45$

Consequently, the required equation of the hyperbola is $\frac{x^2}{36} - \frac{y^2}{45} = 1$

Objective Type Questions

Each of the examples from 11 to 16, has four possible options, out of which one is correct. Choose the correct answer from the given four options (M.C.Q.)

Example 11 The equation of the circle in the first quadrant touching each coordinate axis at a distance of one unit from the origin is:

(A) $x^2 + y^2 - 2x - 2y + 1 = 0$ (B) $x^2 + y^2 - 2x - 2y - 1 = 0$

(C) $x^2 + y^2 - 2x - 2y = 0$ (C) $x^2 + y^2 - 2x + 2y - 1 = 0$

Solution The correct choice is (A), since the equation can be written as $(x - 1)^2 + (y - 1)^2 = 1$ which represents a circle touching both the axes with its centre (1, 1) and radius one unit.

Example 12 The equation of the circle having centre (1, –2) and passing through the point of intersection of the lines $3x + y = 14$ and $2x + 5y = 18$ is

(A) $x^2 + y^2 - 2x + 4y - 20 = 0$ (B) $x^2 + y^2 - 2x - 4y - 20 = 0$

(C) $x^2 + y^2 + 2x - 4y - 20 = 0$ (D) $x^2 + y^2 + 2x + 4y - 20 = 0$

Solution The correct option is (A). The point of intersection of $3x + y - 14 = 0$ and $2x + 5y - 18 = 0$ are $x = 4, y = 2$, i.e., the point (4, 2)

Therefore, the radius is $= \sqrt{9+16} = 5$ and hence the equation of the circle is given by

$$(x - 1)^2 + (y + 2)^2 = 25$$

or $$x^2 + y^2 - 2x + 4y - 20 = 0.$$

Example 13 The area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum is

(A) 12 sq. units

(B) 16 sq. units

(C) 18 sq. units

(D) 24 sq. units

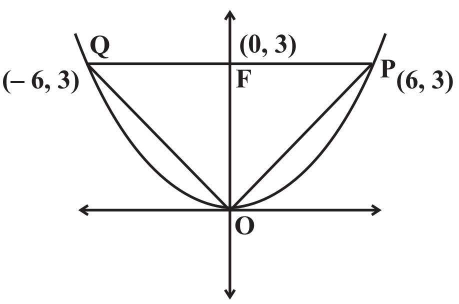


Fig. 11.11

Solution The correct option is (C). From the figure, OPQ represent the triangle whose area is to be determined. The area of the triangle

$= \frac{1}{2}$ PQ × OF $= \frac{1}{2}(12 \times 3) = 18$

Example 14 The equations of the lines joining the vertex of the parabola $y^2 = 6x$ to the points on it which have abscissa 24 are

(A) $y \pm 2x = 0$

(B) $2y \pm x = 0$

(C) $x \pm 2y = 0$

(D) $2x \pm y = 0$

Solution (B) is the correct choice. Let P and Q be points on the parabola $y^2 = 6x$ and OP, OQ be the lines joining the vertex O to the points P and Q whose abscissa are 24.

Thus $\qquad y^2 = 6 \times 24 = 144$

or $\qquad y = \pm 12.$

Therefore the coordinates of the points P and Q are (24, 12) and (24, –12) respectively. Hence the lines are

$$y = \pm \frac{12}{24} x \Rightarrow 2y = \pm x.$$

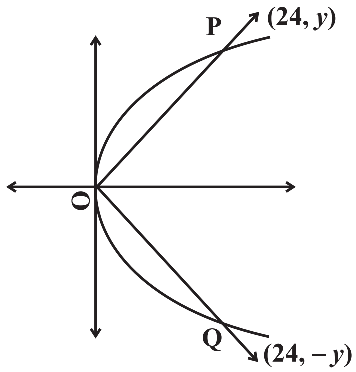


Fig. 11.12

Example 15 The equation of the ellipse whose centre is at the origin and the x-axis, the major axis, which passes through the points (–3, 1) and (2, –2) is

(A) $5x^2 + 3y^2 \;\; 32$

(B) $3x^2 + 5y^2 = 32$

(C) $5x^2 - 3y^2 = 32$

(D) $3x^2 + 5y^2 + 32 = 0$

Solution (B) is the correct choice. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of the ellipse.

Then according to the given conditions, we have

$$\frac{9}{a^2} + \frac{1}{b^2} = 1 \quad \text{and} \quad \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{4}$$

which gives $\qquad a^2 = \frac{32}{3} \quad$ and $\quad b^2 = \frac{32}{5}.$

Hence, required equation of ellipse is $3x^2 + 5y^2 = 32$.

Example 16 The length of the transverse axis along x-axis with centre at origin of a hyperbola is 7 and it passes through the point (5, –2). The equation of the hyperbola is

(A) $\frac{4}{49}x^2 - \frac{196}{51}y^2 = 1$ (B) $\frac{49}{4}x^2 - \frac{51}{196}y^2 = 1$

(C) $\frac{4}{49}x^2 - \frac{51}{196}y^2 = 1$ (D) none of these

Solution (C) is the correct choice. Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represent the hyperbola. Then according to the given condition, the length of transverse axis, i.e., $2a = 7 \Rightarrow a = \frac{7}{2}$.

Also, the point (5, –2) lies on the hyperbola, so, we have

$$\frac{4}{49}(25) - \frac{4}{b^2} = 1 \quad \text{which gives}$$

$b^2 = \frac{196}{51}$. Hence, the equation of the hyperbola is

$$\frac{4}{49}x^2 - \frac{51}{196}y^2 = 1$$

State whether the statements in Examples 17 and 18 are correct or not. Justify.

Example 17 Circle on which the coordinates of any point are (2 + 4 cosθ, –1 + 4 sinθ) where θ is parameter is given by $(x - 2)^2 + (y + 1)^2 = 16$.

Solution True. From given conditions, we have

$$x = 2 + 4\cos\theta \Rightarrow (x - 2) = 4\cos\theta$$

and $$y = -1 + 4\sin\theta \Rightarrow y + 1 = 4\sin\theta.$$

Squaring and adding, we get $(x - 2)^2 + (y + 1)^2 = 16$.

Example 18 A bar of given length moves with its extremities on two fixed straight lines at right angles. Any point of the bar describes an ellipse.

Solution True. Let P (x, y) be any point on the bar such that PA = a and PB = b, clearly from the Fig. 11.13.

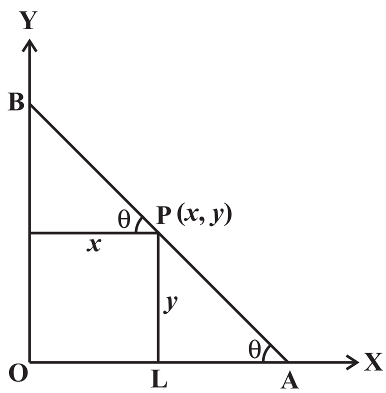


Fig. 11.13

$$x = \text{OL} = b\cos\theta \text{ and}$$
$$y = \text{PL} = a\sin\theta$$

These give $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, which is an ellipse.

Fill in the blanks in Examples 19 to 23.

Example 19 The equation of the circle which passes through the point (4, 5) and has its centre at (2, 2) is ________________ .

Solution As the circle is passing through the point (4, 5) and its centre is (2, 2) so its radius is $\sqrt{(4-2)^2+(5-2)^2} = \sqrt{13}$. Therefore the required answer is $(x-2)^2 + (y-2)^2 = 13$.

Example 20 A circle has radius 3 units and its centre lies on the line $y = x - 1$. If it passes through the point (7, 3), its equation is ________________ .

Solution Let (h, k) be the centre of the circle. Then $k = h - 1$. Therefore, the equation of the circle is given by $(x-h)^2 + [y-(h-1)]^2 = 9$ ... (1)

Given that the circle passes through the point (7, 3) and hence we get

$$(7-h)^2 + (3-(h-1))^2 = 9$$

or $$(7-h)^2 + (4-h)^2 = 9$$

or $$h^2 - 11h + 28 = 0$$

which gives $(h-7)(h-4) = 0 \quad \Rightarrow h = 4 \text{ or } h = 7$

Therefore, the required equations of the circles are $x^2 + y^2 - 8x - 6y + 16 = 0$

or $$x^2 + y^2 - 14x - 12y + 76 = 0$$

Example 21 If the latus rectum of an ellipse with axis along x-axis and centre at origin is 10, distance between foci = length of minor axis, then the equation of the ellipse is ________________ .

Solution Given that $\dfrac{2b^2}{a} = 10$ and $2ae = 2b \Rightarrow b = ae$

Again, we know that

$$b^2 = a^2(1-e^2)$$

or $$2a^2e^2 = a^2 \Rightarrow e = \frac{1}{\sqrt{2}} \quad (\text{using } b = ae)$$

Thus $$a = b\sqrt{2}$$

Again $\qquad \dfrac{2b^2}{a} = 10$

or $\qquad b = 5\sqrt{2}$. $\quad$ Thus we get $a = 10$

Therefore, the required equation of the ellipse is

$$\frac{x^2}{100} + \frac{y^2}{50} = 1$$

Example 22 The equation of the parabola whose focus is the point (2, 3) and directrix is the line $x - 4y + 3 = 0$ is ________________ .

Solution Using the definition of parabola, we have

$$\sqrt{(x-2)^2 + (y-3)^2} = \left|\frac{x-4y+3}{\sqrt{17}}\right|$$

Squaring, we get

$$17\,(x^2 + y^2 - 4x - 6y + 13) = x^2 + 16y^2 + 9 - 8xy - 24y + 6x$$

or $\quad 16x^2 + y^2 + 8xy - 74x - 78y + 212 = 0$

Example 23 The eccentricity of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ which passes through the points (3, 0) and $(3\sqrt{2}, 2)$ is ________________ .

Solution Given that the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is passing through the points (3, 0) and $(3\sqrt{2}, 2)$, so we get $a^2 = 9$ and $b^2 = 4$.

Again, we know that $b^2 = a^2\,(e^2 - 1)$. This gives

$$4 = 9\,(e^2 - 1)$$

or $\qquad e^2 = \dfrac{13}{9}$

or $\qquad e = \dfrac{\sqrt{13}}{3}$.

11.3 EXERCISE

Short Answer Type

1. Find the equation of the circle which touches the both axes in first quadrant and whose radius is a.

2. Show that the point (x, y) given by $x = \dfrac{2at}{1+t^2}$ and $y = \dfrac{a(1-t^2)}{1+t^2}$ lies on a circle for all real values of t such that $-1 \leq t \leq 1$ where a is any given real numbers.

3. If a circle passes through the point (0, 0) $(a, 0)$, $(0, b)$ then find the coordinates of its centre.

4. Find the equation of the circle which touches x-axis and whose centre is (1, 2).

5. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then find the radius of the circle.

   [**Hint:** Distance between given parallel lines gives the diameter of the circle.]

6. Find the equation of a circle which touches both the axes and the line $3x - 4y + 8 = 0$ and lies in the third quadrant.

   [**Hint:** Let a be the radius of the circle, then $(-a, -a)$ will be centre and perpendicular distance from the centre to the given line gives the radius of the circle.]

7. If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (3, 4), then find the coordinate of the other end of the diameter.

8. Find the equation of the circle having (1, –2) as its centre and passing through $3x + y = 14$, $2x + 5y = 18$

9. If the line $y = \sqrt{3}\, x + k$ touches the circle $x^2 + y^2 = 16$, then find the value of k.

   [**Hint:** Equate perpendicular distance from the centre of the circle to its radius].

10. Find the equation of a circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has double of its area.

    [**Hint:** concentric circles have the same centre.]

11. If the latus rectum of an ellipse is equal to half of minor axis, then find its eccentricity.

12. Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the eccentricity and foci.

13. If the eccentricity of an ellipse is $\dfrac{5}{8}$ and the distance between its foci is 10, then find latus rectum of the ellipse.

14. Find the equation of ellipse whose eccentricity is $\frac{2}{3}$, latus rectum is 5 and the centre is (0, 0).

15. Find the distance between the directrices of the ellipse $\frac{x^2}{36}+\frac{y^2}{20}=1$.

16. Find the coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4.

17. Find the length of the line-segment joining the vertex of the parabola $y^2 = 4ax$ and a point on the parabola where the line-segment makes an angle θ to the x-axis.

18. If the points (0, 4) and (0, 2) are respectively the vertex and focus of a parabola, then find the equation of the parabola.

19. If the line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$ then find the value of m.

[**Hint:** Solving the equation of line and parabola, we obtain a quadratic equation and then apply the tangency condition giving the value of m].

20. If the distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then obtain the equation of the hyperbola.

21. Find the eccentricity of the hyperbola $9y^2 - 4x^2 = 36$.

22. Find the equation of the hyperbola with eccentricity $\frac{3}{2}$ and foci at $(\pm 2, 0)$.

Long Answer Type

23. If the lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 square units, then obtain the equation of the circle.

24. Find the equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$.

25. Find the equation of a circle whose centre is (3, –1) and which cuts off a chord of length 6 units on the line $2x - 5y + 18 = 0$.

[**Hint:** To determine the radius of the circle, find the perpendicular distance from the centre to the given line.]

26. Find the equation of a circle of radius 5 which is touching another circle

$x^2 + y^2 - 2x - 4y - 20 = 0$ at (5, 5).

27. Find the equation of a circle passing through the point (7, 3) having radius 3 units and whose centre lies on the line $y = x - 1$.

28. Find the equation of each of the following parabolas

(a) Directrix $x = 0$, focus at (6, 0) (b) Vertex at (0, 4), focus at (0, 2)

(c) Focus at (–1, –2), directrix $x - 2y + 3 = 0$

29. Find the equation of the set of all points the sum of whose distances from the points (3, 0) and (9, 0) is 12.

30. Find the equation of the set of all points whose distance from (0, 4) are $\frac{2}{3}$ of their distance from the line $y = 9$.

31. Show that the set of all points such that the difference of their distances from (4, 0) and (– 4, 0) is always equal to 2 represent a hyperbola.

32. Find the equation of the hyperbola with

(a) Vertices (± 5, 0), foci (± 7, 0) (b) Vertices (0, ± 7), $e = \frac{4}{3}$

(c) Foci (0, ± $\sqrt{10}$), passing through (2, 3)

Objective Type Questions

State Whether the statements in each of the Exercises from 33 to 40 are True or False. Justify

33. The line $x + 3y = 0$ is a diameter of the circle $x^2 + y^2 + 6x + 2y = 0$.

34. The shortest distance from the point (2, –7) to the circle $x^2 + y^2 - 14x - 10y - 151 = 0$ is equal to 5.

[**Hint:** The shortest distance is equal to the difference of the radius and the distance between the centre and the given point.]

35. If the line $lx + my = 1$ is a tangent to the circle $x^2 + y^2 = a^2$, then the point (l, m) lies on a circle.

[**Hint:** Use that distance from the centre of the circle to the given line is equal to radius of the circle.]

36. The point (1, 2) lies inside the circle $x^2 + y^2 - 2x + 6y + 1 = 0$.

37. The line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$ if $ln = am^2$.

38. If P is a point on the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ whose foci are S and S′, then PS + PS′ = 8.

39. The line $2x + 3y = 12$ touches the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 2$ at the point (3, 2).

40. The locus of the point of intersection of lines $\sqrt{3}x - y - 4\sqrt{3}k = 0$ and

$\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different value of k is a hyperbola whose eccentricity is 2.

[**Hint:** Eliminate k between the given equations]

Fill in the Blank in Exercises from 41 to 46.

41. The equation of the circle having centre at (3, – 4) and touching the line $5x + 12y - 12 = 0$ is ________________ .

[**Hint:** To determine radius find the perpendicular distance from the centre of the circle to the line.]

42. The equation of the circle circumscribing the triangle whose sides are the lines $y = x + 2$, $3y = 4x$, $2y = 3x$ is ________________ .

43. An ellipse is described by using an endless string which is passed over two pins. If the axes are 6 cm and 4 cm, the length of the string and distance between the pins are _____________.

44. The equation of the ellipse having foci (0, 1), (0, –1) and minor axis of length 1 is ________________ .

45. The equation of the parabola having focus at (–1, –2) and the directrix $x - 2y + 3 = 0$ is ________________ .

46. The equation of the hyperbola with vertices at (0, ± 6) and eccentricity $\frac{5}{3}$ is ______________ and its foci are ________________ .

Choose the correct answer out of the given four options (M.C.Q.) in Exercises 47 to 59.

47. The area of the circle centred at (1, 2) and passing through (4, 6) is

(A) 5π (B) 10π (C) 25π (D) none of these

48. Equation of a circle which passes through (3, 6) and touches the axes is

(A) $x^2 + y^2 + 6x + 6y + 3 = 0$ (B) $x^2 + y^2 - 6x - 6y - 9 = 0$

(C) $x^2 + y^2 - 6x - 6y + 9 = 0$ (D) none of these

49. Equation of the circle with centre on the y-axis and passing through the origin and the point (2, 3) is

(A) $x^2 + y^2 + 13y = 0$ (B) $3x^2 + 3y^2 + 13x + 3 = 0$

(C) $6x^2 + 6y^2 - 13x = 0$ (D) $x^2 + y^2 + 13x + 3 = 0$

50. The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length $3a$ is

(A) $x^2 + y^2 = 9a^2$ (B) $x^2 + y^2 = 16a^2$

(C) $x^2 + y^2 = 4a^2$ (D) $x^2 + y^2 = a^2$

[**Hint:** Centroid of the triangle coincides with the centre of the circle and the radius of the circle is $\frac{2}{3}$ of the length of the median]

51. If the focus of a parabola is (0, –3) and its directrix is $y = 3$, then its equation is

(A) $x^2 = -12y$ (B) $x^2 = 12y$ (C) $y^2 = -12x$ (D) $y^2 = 12x$

52. If the parabola $y^2 = 4ax$ passes through the point (3, 2), then the length of its latus rectum is

(A) $\frac{2}{3}$ (B) $\frac{4}{3}$ (C) $\frac{1}{3}$ (D) 4

53. If the vertex of the parabola is the point (–3, 0) and the directrix is the line $x + 5 = 0$, then its equation is

(A) $y^2 = 8(x + 3)$ (B) $x^2 = 8(y + 3)$

(C) $y^2 = -8(x + 3)$ (D) $y^2 = 8(x + 5)$

54. The equation of the ellipse whose focus is (1, –1), the directrix the line $x - y - 3 = 0$ and eccentricity $\frac{1}{2}$ is

(A) $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$

(B) $7x^2 + 2xy + 7y^2 + 7 = 0$

(C) $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$

(D) none

55. The length of the latus rectum of the ellipse $3x^2 + y^2 = 12$ is

(A) 4 (B) 3 (C) 8 (D) $\frac{4}{\sqrt{3}}$

56. If e is the eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a < b)$, then

(A) $b^2 = a^2(1 - e^2)$ (B) $a^2 = b^2(1 - e^2)$

(C) $a^2 = b^2(e^2 - 1)$ (D) $b^2 = a^2(e^2 - 1)$

57. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half of the distance between the foci is

(A) $\frac{4}{3}$ (B) $\frac{4}{\sqrt{3}}$ (C) $\frac{2}{\sqrt{3}}$ (D) none of these

58. The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$. Its equation is

(A) $x^2 - y^2 = 32$ (B) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ (C) $2x - 3y^2 = 7$ (D) none of these

59. Equation of the hyperbola with eccentricty $\frac{3}{2}$ and foci at $(\pm 2, 0)$ is

(A) $\frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$ (B) $\frac{x^2}{9} - \frac{y^2}{9} = \frac{4}{9}$ (C) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ (D) none of these

Chapter 12

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

12.1 Overview

12.1.1 *Coordinate axes and coordinate planes* Let X′OX, Y′OY, Z′OZ be three mutually perpendicular lines that pass through a point O such that X′OX and Y′OY lies in the plane of the paper and line Z′OZ is perpendicular to the plane of paper. These three lines are called rectangular axes (lines X′OX, Y′OY and Z′OZ are called x-axis, y-axis and z-axis). We call this coordinate system a three dimensional space, or simply space.

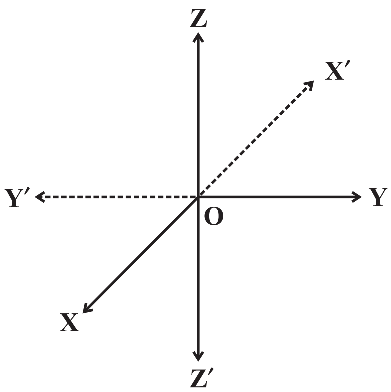


Fig. 12.1

The three axes taken together in pairs determine xy, yz, zx-plane , i.e., three coordinate planes. Each plane divide the space in two parts and the three coordinate planes together divide the space into eight regions (parts) called octant, namely (i) OXYZ (ii) OX′YZ (iii) OXY′Z (iv) OXYZ′ (v) OXY′Z′ (vi) OX′YZ′ (vii) OX′Y′Z (viii) OX′Y′Z′. (Fig.12.1).

Let P be any point in the space, not in a coordinate plane, and through P pass planes parallel to the coordinate planes yz, zx and xy meeting the coordinate axes in the points A, B, C respectively.

Three planes are

(i) ADPF || yz-plane (ii) BDPE || xz-plane

(iii) CFPE || xy-plane

These planes determine a rectangular parallelopiped which has three pairs of rectangular faces

(A D P F, O B E C),(B D P E, C F A O) and (A O B D, FPEC) (Fig 12.2)

12.1.2 *Coordinate of a point in space* An arbitrary point P in three-dimensional space is assigned coordinates (x_0, y_0, z_0) provided that

(1) the plane through P parallel to the *yz*-plane intersects the *x*-axis at $(x_0, 0, 0)$;

(2) the plane through P parallel to the *xz*-plane intersects the *y*-axis at $(0, y_0, 0)$;

(3) the plane through P parallel to the *xy*-plane intersects the *z*-axis at $(0, 0, z_0)$.

The space coordinates (x_0, y_0, z_0) are called the Cartesian coordinates of P or simply the rectangular coordinates of P.

Moreover we can say, the plane ADPF (Fig.12.2) is perpendicular to the *x*-axis or *x*-axis is perpendicular to the plane ADPF and hence perpendicular to every line in the plane. Therefore, PA is perpendicular to OX and OX is perpendicular to PA. Thus A is the foot of perpendicular drawn from P on *x*-axis and distance of this foot A from O is *x*-coordinate of point P. Similarly, we call B and C are the feet of perpendiculars drawn from point P on the *y* and *z*-axis and distances of these feet B and C from O are the *y* and *z* coordinates of the point P.

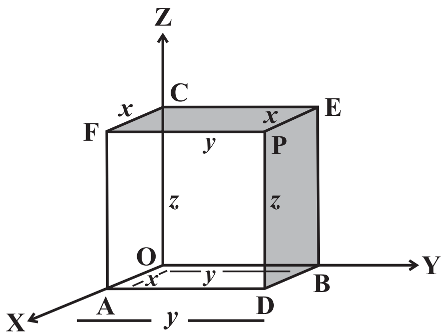


Fig. 12.2

Hence the coordinates *x*, *y* *z* of a point P are the perpendicular distance of P from the three coordinate planes *yz*, *zx* and *xy*, respectively.

12.1.3 ***Sign of coordinates of a point*** The distance measured along or parallel to OX, OY, OZ will be positive and distance moved along or parallel to OX′, OY′, OZ′ will be negative. The three mutually perpendicular coordinate plane which in turn divide the space into eight parts and each part is know as octant. The sign of the coordinates of a point depend upon the octant in which it lies. In first octant all the coordinates are positive and in seventh octant all coordinates are negative. In third octant *x*, *y* coordinates are negative and *z* is positive. In fifth octant *x*, *y* are positive and *z* is negative. In fourth octant *x*, *z* are positive and *y* is negative. In sixth octant *x*, *z* are negative *y* is positive. In the second octant *x* is negative and *y* and *z* are positive.

Octants → Coordinates ↓	I OXYZ	II OX′YZ	III OX′Y′Z	IV OXY′Z	V OXYZ′	VI OX′YZ′	VII OX′Y′Z′	VIII OXY′Z′
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

12.1.4 *Distance formula* The distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

A paralleopiped is formed by planes drawn through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) parallel to the coordinate planes. The length of edges are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ and length of diagonal is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

12.1.5 *Section formula* The coordinates of the point R which divides the line segment joining two points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) internally or externally in the ratio $m : n$ are given by $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right), \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$, respectively.

The coordinates of the mid-point of the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

The coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and x_3, y_3, z_3 are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.

12.2 Solved Examples

Short Answer Type

Example 1 Locate the points (i) (2, 3, 4) (ii) (–2, –2, 3) in space.

Solution

(i) To locate the point (2, 3, 4) in space, we move 2 units from O along the positive direction of x-axis. Let this point be A (2, 0, 0). From the point A moves 3 units parallel to +ve direction of y-axis.Let this point be B (2, 3, 0). From the point B moves 4 units along positive direction of z-axis. Let this point be P (2, 3, 4) Fig.(12.3).

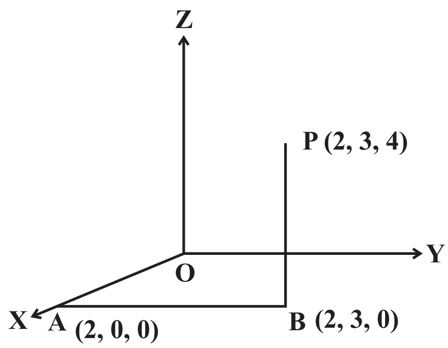


Fig. 12.3

(ii) From the origin, move 2 units along the negative direction of x-axis. Let this point be A (–2, 0, 0). From the point A move 2 units parallel to negative direction of y-axis.

Let this point be B (–2, –2, 0). From B move 3 units parallel to positive direction of z - axis. This is our required point Q (–2, –2, 3) (Fig.12.4.)

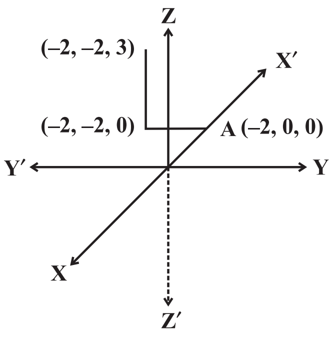


Fig. 12.4

Example 2 Sketch the plane (i) $x = 1$ (ii) $y = 3$ (iii) $z = 4$

Solution

(i) The equation of the plan $x = 0$ represents the yz-plane and equation of the plane $x = 1$ represents the plane parallel to yz-plane at a distance 1 unit above yz-plane. Now, we draw a plane parallel to yz- plane at a distance 1 unit above yz-plane Fig.12.5(a).

(ii) The equation of the plane $y = 0$ represents the xz plane and the equation of the plane $y = 3$ represents the plane parallel to xz plane at a distance 3 unit above xz plane (Fig. 12.5(b)).

(iii) The equation of the plane $z = 0$ represents the xy-plane and $z = 3$ represents the plane parallel to xy-plane at a distance 3 unit above xy-plane (Fig. 12.5(c)).

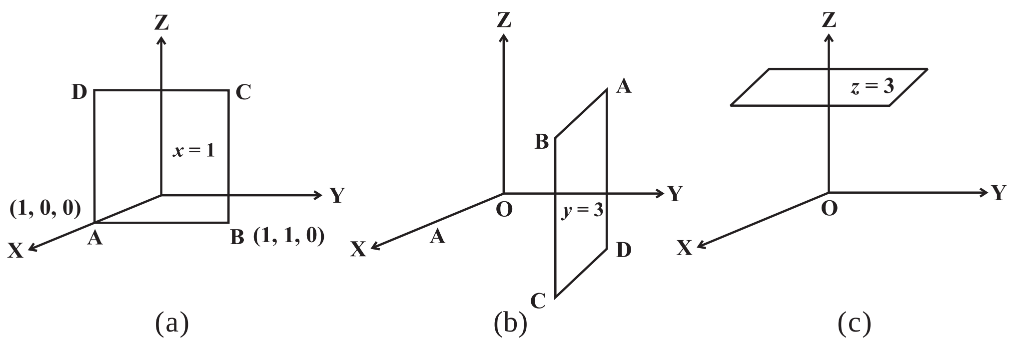


Fig. 12.5

Example 3 Let L, M, N be the feet of the perpendiculars drawn from a point P (3, 4, 5) on the *x*, *y* and *z*-axes respectively. Find the coordinates of L, M and N.

Solution Since L is the foot of perpendicular from P on the *x*-axis, its *y* and *z* co-ordinates are zero. The coordinates of L is (3, 0, 0). Similarly, the coordinates of M and N are (0, 4, 0) and (0, 0, 5), respectively.

Example 4 Let L, M, N be the feet of the perpendicular segments drawn from a point P (3, 4, 5) on the *xy*, *yz* and *zx*-planes, respectively. What are the coordinates of L, M and N?

Solution Since L is the foot of perpendicular segment from P on the *xy*-plane, *z*-coordinate is zero in the *xy*-plane. Hence, coordinates of L is (3, 4, 0). Similarly, we can find the coordinates of of M (0, 4, 5) and N (3, 0, 5), Fig.12.6.

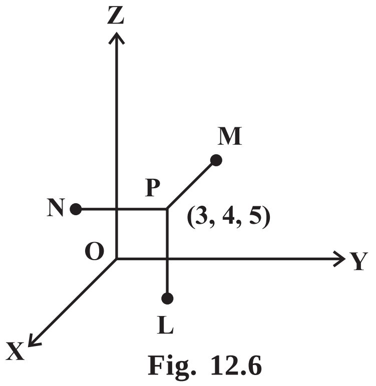


Fig. 12.6

Example 5 Let L, M, N are the feet of the perpendiculars drawn from the point P (3, 4, 5) on the *xy*, *yz* and *zx*-planes, respectively. Find the distance of these points L, M, N from the point P, Fig.12.7.

Solution L is the foot of perpendicular drawn from the point P (3, 4, 5) to the *xy*-plane. Therefore, the coordinate of the point L is (3, 4, 0). The distance between the point (3, 4, 5) and (3, 4, 0) is 5. Similarly, we can find the lengths of the foot of perpendiculars on *yz* and *zx*-plane which are 3 and 4 units, respectively.

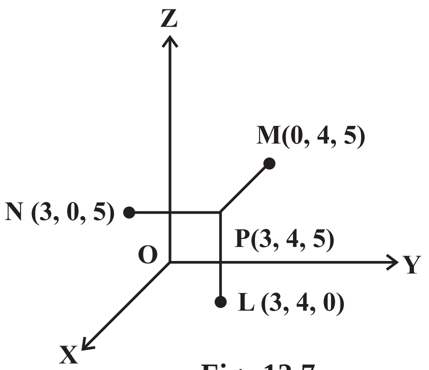


Fig. 12.7

Example 6 Using distance formula show that the points P (2, 4, 6), Q (– 2, – 2, – 2) and R (6, 10, 14) are collinear.

Solution Three points are collinear if the sum of any two distances is equal to the third distance.

$$PQ = \sqrt{(-2-2)^2+(-2-4)^2+(-2-6)^2} = \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$$

$$QR = \sqrt{(6+2)^2+(10+2)^2+(14+2)^2} = \sqrt{64+144+256} = \sqrt{464} = 4\sqrt{29}$$

$$PR = \sqrt{(6-2)^2+(10-4)^2+(14-6)^2} = \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$$

Since QR = PQ + PR. Therefore, the given points are collinear.

Example 7 Find the coordinates of a point equidistant from the four points O (0, 0, 0), A (l, 0, 0), B (0, m, 0) and C (0, 0, n).

Solution Let P (x, y, z) be the required point. Then OP = PA = PB = PC.

Now OP = PA $\Rightarrow \text{OP}^2 = \text{PA}^2 \Rightarrow x^2 + y^2 + z^2 = (x - l)^2 + (y - 0)^2 + (z - 0)^2 \Rightarrow x = \frac{l}{2}$

Similarly, OP = PB $\Rightarrow y = \frac{m}{2}$ and OP = PC $\Rightarrow z = \frac{n}{2}$

Hence, the coordinate of the required point are $\left(\frac{l}{2}, \frac{m}{2}, \frac{n}{2}\right)$.

Example 8 Find the point on x-axis which is equidistant from the point A (3, 2, 2) and B (5, 5, 4).

Solution The point on the x-axis is of form P (x, 0, 0). Since the points A and B are equidistant from P. Therefore $\text{PA}^2 = \text{PB}^2$, i.e.,

$(x - 3)^2 + (0 - 2)^2 + (0 - 2)^2 = (x - 5)^2 + (0 - 5)^2 + (0 - 4)^2$

$\Rightarrow 4x = 25 + 25 + 16 - 17$ i.e., $x = \frac{49}{4}$.

Thus, the point P on the x - axis is $\left(\frac{49}{4}, 0, 0\right)$ which is equidistant from A and B.

Example 9 Find the point on y-axis which is at a distance $\sqrt{10}$ from the point (1, 2, 3)

Solution Let the point P be on y-axis. Therefore, it is of the form P (0, y, 0).

The point (1, 2, 3) is at a distance $\sqrt{10}$ from (0, y, 0). Therefore

$$\sqrt{(1-0)^2 + (2-y)^2 + (3-0)^2} = \sqrt{10}$$

$$\Rightarrow y^2 - 4y + 4 = 0 \Rightarrow (y - 2)^2 = 0 \Rightarrow y = 2$$

Hence, the required point is (0, 2, 0).

Example 10 If a parallelopiped is formed by planes drawn through the points (2, 3, 5) and (5, 9, 7) parallel to the coordinate planes, then find the length of edges of a parallelopiped and length of the diagonal.

Solution Length of edges of the parallelopiped are 5 – 2, 9 – 3, 7 – 5 i.e., 3, 6, 2.

Length of diagonal is $\sqrt{3^2 + 6^2 + 2^2} = 7$ units.

Example 11 Show that the points (0, 7, 10), (–1, 6, 6) and (– 4, 9, 6) form a right angled isosceles triangle.

Solution Let P (0, 7, 10), Q (–1, 6, 6) and R (– 4, 9, 6) be the given three points.

Here $PQ = \sqrt{1+1+16} = 3\sqrt{2}$

$QR = \sqrt{9+9+0} = 3\sqrt{2}$

$PR = \sqrt{16+4+16} = 6$

Now $PQ^2 + QR^2 = (3\sqrt{2})^2 + (3\sqrt{2})^2 = 18 + 18 = 36 = (PR)^2$

Therefore, Δ PQR is a right angled triangle at Q. Also PQ = QR. Hence Δ PQR is an isosceles triangle.

Example 12 Show that the points (5, –1, 1), (7, – 4, 7), (1 – 6, 10) and (–1, – 3, 4) are the vertices of a rhombus.

Solution Let A (5, – 1, 1), B (7, – 4, 7), C(1, – 6, 10) and D (– 1, – 3, 4) be the four points of a quadrilateral. Here

$AB = \sqrt{4+9+36} = 7$, $BC = \sqrt{36+4+9} = 7$, $CD = \sqrt{4+9+36} = 7$,

$DA = \sqrt{23+4+9} = 7$

Note that AB = BC = CD = DA. Therefore, ABCD is a rhombus.

Example 13 Find the ratio in which the line segment joining the points (2, 4, 5) and (3, 5, – 4) is divided by the *xz*-plane.

Solution Let the joint of P (2, 4, 5) and Q (3, 5, – 4) be divided by *xz*-plane in the ratio k:1 at the point R(x, y, z). Therefore

$$x = \frac{3k+2}{k+1},\ y = \frac{5k+4}{k+1},\ z = \frac{-4k+5}{k+1}$$

Since the point R (x, y, z) lies on the *xz*-plane, the *y*-coordinate should be zero,i.e.,

$$\frac{5k+4}{k+1} = 0 \Rightarrow k = -\frac{4}{5}$$

Hence, the required ratio is – 4 : 5, i.e.; externally in the ratio 4 : 5.

Example 14 Find the coordinate of the point P which is five - sixth of the way from A (– 2, 0, 6) to B (10, – 6, – 12).

Solution Let P (x, y, z) be the required point, i.e., P divides AB in the ratio 5 : 1. Then

$$\text{P}(x, y, z) = \left(\frac{5\times10+1\times-2}{5+1}, \frac{5\times-6+1\times0}{5+1}, \frac{5\times-12+1\times6}{5+1}\right) = (8, -5, -9)$$

Example 15 Describe the vertices and edges of the rectangular parallelopiped with vertex (3, 5, 6) placed in the first octant with one vertex at origin and edges of parallelopiped lie along x, y and z-axes.

Solution The six planes of the parallelopiped are as follows:

Plane OABC lies in the xy-plane. The z-coordinate of every point in this plane is zero. $z = 0$ is the equation of this xy-plane. Plane PDEF is parallel to xy-plane and 6 unit distance above it. The equation of the plane is $z = 6$. Plane ABPF represents plane $x = 3$. Plane OCDE lies in the yz-plane and $x = 0$ is the equation of this plane. Plane AOEF lies in the xz-plane. The y coordinate of everypoint in this plane is zero. Therefore, $y = 0$ is the equation of plane.

Plane BCDP is parallel to the plane AOEF at a distance $y = 5$.

Edge OA lies on the x-axis. The x-axis has equation $y = 0$ and $z = 0$.

Edges OC and OE lie on y-axis and z-axis, respectively. The y-axis has its equation $z = 0, x = 0$. The z-axis has its equation $x = 0$, $y = 0$. The perpendicular distance of the point P (3, 5, 6) from the x-axis is $\sqrt{5^2+6^2} = \sqrt{61}$.

The perpendicular distance of the point P (3, 5, 6) from y-axis and z-axis are $\sqrt{3^2+6^2} = \sqrt{45}$ and $\sqrt{3^2+5^2}$ =, respectively.

The coordinates of the feet of perpendiculars from the point P (3, 5, 6) to the coordinate axes are A, C, E. The coordinates of feet of perpendiculars from the point P on the coordinate planes xy, yz and zx are (3, 5, 0), (0, 5, 6) and (3, 0, 6). Also, perpendicular

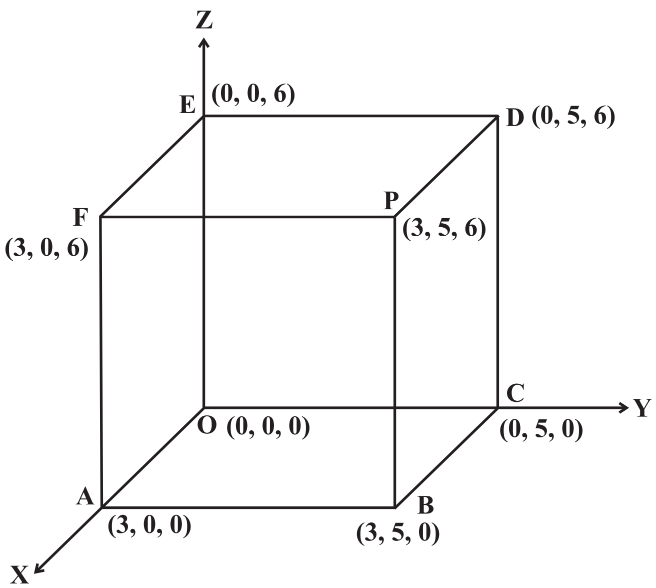


Fig. 12.8

distance of the point P from the *xy*, *yz* and *zx*-planes are 6, 5 and 3, respectively, Fig.12.8.

Example 16 Let A (3, 2, 0), B (5, 3, 2), C (– 9, 6, – 3) be three points forming a triangle. AD, the bisector of $\angle$BAC, meets BC in D. Find the coordinates of the point D.

Solution Note that

$$AB=\sqrt{(5-3)^2+(3-2)^2+(2-0)^2}=\sqrt{4+1+4}=3$$

$$AC=\sqrt{(-9-3)^2+(6-2)^2+(-3-0)^2}=\sqrt{144+16+9}=13$$

Since AD is the bisector of $\angle$ BAC,We have $\frac{BD}{DC}=\frac{AB}{AC}=\frac{3}{13}$

i.e., D divides BC in the ratio 3 : 13. Hence, the coordinates of D are

$$\left(\frac{3(-9)+13(5)}{3+13},\frac{3(6)+13(3)}{3+13},\frac{3(-3)+13(2)}{3+13}\right)=\left(\frac{19}{8},\frac{57}{16},\frac{17}{16}\right)$$

Example 17 Determine the point in *yz*-plane which is equidistant from three points A (2, 0 3) B (0, 3, 2) and C (0, 0, 1).

Solution Since *x*-coordinate of every point in *yz*-plane is zero. Let P (0, *y*, *z*) be a point on the *yz*-plane such that PA = PB = PC. Now

PA = PB $\Rightarrow (0-2)^2+(y-0)^2+(z-3)^2=(0-0)^2+(y-3)^2+(z-2)^2$, i.e. $z-3y=0$

and PB = PC

$\Rightarrow y^2+9-6y+z^2+4-4z=y^2+z^2+1-2z$, i.e. $3y+z=6$

Simplifying the two equating, we get $y = 1$, $z = 3$

Here, the coordinate of the point P are (0, 1, 3).

Objective Type Questions

Choose the correct answer out of given four options in each of the Examples from 18 to 23 (M.C.Q.).

Example 18 The length of the foot of perpendicular drawn from the point P (3, 4, 5) on *y*-axis is

(A) 10 (B) $\sqrt{34}$ (C) $\sqrt{113}$ (D) $5\sqrt{2}$

Solution Let *l* be the foot of perpendicular from point P on the *y*-axis. Therefore, its *x* and *z*-coordinates are zero, i.e., (0, 4, 0). Therefore, distance between the points (0, 4, 0) and (3, 4, 5) is $\sqrt{9+25}$ i.e., $\sqrt{34}$.

Example 19 What is the perpendicular distance of the point P (6, 7, 8) from *xy*-plane?

(A) 8 (B) 7 (C) 6 (D) None of these

Solution Let L be the foot of perpendicular drawn from the point P (6, 7, 8) to the *xy*-plane and the distance of this foot L from P is *z*-coordinate of P, i.e., 8 units.

Example 20 L is the foot of the perpendicular drawn from a point P (6, 7, 8) on the *xy*-plane. What are the coordinates of point L?

(A) (6, 0, 0) (B) (6, 7, 0) (C) (6, 0, 8) (D) none of these

Solution Since L is the foot of perpendicular from P on the *xy*-plane, *z*-coordinate is zero in the *xy*-plane. Hence, coordinates of L are (6, 7, 0).

Example 21 L is the foot of the perpendicular drawn from a point (6, 7, 8) on *x*-axis. The coordinates of L are

(A) (6, 0, 0) (B) (0, 7, 0) (C) (0, 0, 8) (D) none of these

Solution Since L is the foot of perpendicular from P on the *x*- axis, *y* and *z*-coordinates are zero. Hence, the coordinates of L are (6, 0, 0).

Example 22 What is the locus of a point for which $y = 0$, $z = 0$?

(A) equation of *x*-axis (B) equation of *y*-axis

(C) equation of *z*-axis (D) none of these

Solution Locus of the point $y = 0, z = 0$ is *x*-axis, since on *x*-axis both $y = 0$ and $z = 0$.

Example 23 L, is the foot of the perpendicular drawn from a point P (3, 4, 5) on the *xz* plane. What are the coordinates of point L ?

(A) (3, 0, 0) (B) (0, 4, 5) (C) (3, 0, 5) (D) (3, 4, 0)

Solution Since L is the foot of perpendicular segment drawn from the point P (3, 4, 5) on the *xz*-plane. Since the *y*-coordinates of all points in the *xz*-plane are zero, coordinate of the foot of perpendicular are (3, 0, 5).

Fill in the blanks in Examples 24 to 28.

Example 24 A line is parallel to *xy*-plane if all the points on the line have equal ______.

Solution A line parallel to *xy*-plane if all the points on the line have equal *z*-coordinates.

Example 25 The equation $x = b$ represents a plane parallel to ______ plane.

Solution Since $x = 0$ represent *yz*-plane, therefore $x = b$ represent a plane parallel to *yz* -plane at a unit distance *b* from the origin.

Example 26 Perpendicular distance of the point P (3, 5, 6) from *y*-axis is ________

Solution Since M is the foot of perpendicular from P on the y-axis, therefore, its x and z-coordinates are zero. The coordinates of M is (0, 5, 0). Therefore, the perpendicular distance of the point P from y-axis $\sqrt{3^2+6^2} = \sqrt{45}$.

Example 27 L is the foot of perpendicular drawn from the point P (3, 4, 5) on zx-planes. The coordinates of L are ________.

Solution Since L is the foot of perpendicular from P on the zx-plane, y-coordinate of every point is zero in the zx-plane. Hence, coordinate of L are (3, 0, 5).

Example 28 The length of the foot of perpendicular drawn from the point P (a, b, c) on z-axis is ______.

Solution The coordinates of the foot of perpendicular from the point P (a, b, c) on z-axis is (0, 0,c). The distance between the point P (a, b, c) and (0, 0, c) is $\sqrt{a^2+b^2}$.

Check whether the statements in Example from 30 to 37 are True or False

Example 29 The y-axis and z-axis, together determine a plane known as yz-plane.

Solution True

Example 30 The point (4, 5, – 6) lies in the VI$^{\text{th}}$ octant.

Solution False, the point (4, 5, – 6) lies in the V$^{\text{th}}$ octant,

Example 31 The x-axis is the intersection of two planes xy-plane and xz plane.

Solution True.

Example 32 Three mutually perpendicular planes divide the space into 8 octants.

Solution True.

Example 33 The equation of the plane $z = 6$ represent a plane parallel to the xy-plane, having a z-intercept of 6 units.

Solution True.

Example 34 The equation of the plane $x = 0$ represent the yz-plane.

Solution True.

Example 35 The point on the x-axis with x-coordinate equal to x_0 is written as (x_0, 0, 0).

Solution True.

Example 36 $x = x_0$ represent a plane parallel to the yz-plane.

Solution True.

Match each item given under the column C_1 to its correct answer given under column C_2.

Example 37

Column C_1		Column C_2	
(a)	If the centriod of the triangle is origin and two of its vertices are (3, – 5, 7) and (–1, 7, – 6) then the third vertex is	(i)	Parallelogram
(b)	If the mid-points of the sides of triangle are (1, 2, – 3), (3, 0, 1) and (–1, 1, – 4) then the centriod is	(ii)	(–2, –2, –1)
(c)	The points (3, – 1, – 1), (5, – 4, 0), (2, 3, – 2) and (0, 6, – 3) are the vertices of a	(iii)	as Isosceles right-angled triangle
(d)	Point A(1, –1, 3), B (2, – 4, 5) and C (5, – 13, 11) are	(iv)	(1, 1, – 2)
(e)	Points A (2, 4, 3), B (4, 1, 9) and C (10, – 1, 6) are the vertices of	(v)	Collinear

Solution (a)

Let A (3, – 5, 7), B (– 1, 7, – 6), C (x, y, z) be the vertices of a Δ ABC with centriod (0, 0, 0)

Therefore, (0, 0, 0) = $\left(\frac{3-1+x}{3}, \frac{-5+7+y}{3}, \frac{7-6+z}{3}\right)$. This implies $\frac{x+2}{3}=0$, $\frac{y+2}{3}=0$, $\frac{z+1}{3}=0$.

Hence x = – 2, y = – 2, and z = – 1.Therefore (a) ↔ (ii)

(b) Let ABC be the given Δ and DEF be the mid-points of the sides BC, CA, AB, respectively. We know that the centriod of the Δ ABC = centriod of Δ DEF.

Therefore, centriod of Δ DEF is $\left(\frac{1+3-1}{3}, \frac{2+0+1}{3}, \frac{-3+1-4}{3}\right)$ = (1, 1, – 2)

Hence (b) $\leftrightarrow$ (iv)

(c) Mid-point of diagonal AC is $\left(\frac{3+2}{2}, \frac{-1+3}{2}, \frac{-1-2}{2}\right) = \left(\frac{5}{2}, 1, \frac{-3}{2}\right)$

Mid-point of diagonal BD is $\left(\frac{5+0}{2}, \frac{-4+6}{2}, \frac{0-3}{2}\right) = \left(\frac{5}{2}, 1, \frac{-3}{2}\right)$

Diagonals of parallelogram bisect each other. Therefore (c) $\leftrightarrow$ (i)

(d) $|AB| = \sqrt{(2-1)^2 + (-4+1)^2 + (5-3)^2} = \sqrt{14}$

$|BC| = \sqrt{(5-2)^2 + (-13+4)^2 + (11-5)^2} = 3\sqrt{14}$

$|AC| = \sqrt{(5-1)^2 + (-13+1)^2 + (11-3)^2} = 4\sqrt{14}$

Now $|AB| + |BC| = |AC|$. Hence Points A, B, C are collinear. Hence (d) $\leftrightarrow$ (v)

(e) AB = $\sqrt{4+9+36} = 7$

BC = $\sqrt{36+4+9} = 7$

CA = $\sqrt{64+25+9} = 7\sqrt{2}$

Now $AB^2 + BC^2 = AC^2$. Hence ABC is an isosceles right angled triangle and hence (e) $\leftrightarrow$ (iii)

12.3 EXERCISE

Short Answer Type

1. Locate the following points:

(i) (1, – 1, 3), (ii) (– 1, 2, 4)

(iii) (– 2, – 4, –7) (iv) (– 4, 2, – 5).

2. Name the octant in which each of the following points lies.

(i) (1, 2, 3), (ii) (4, – 2, 3), (iii) (4, –2, –5) (iv) (4, 2, –5)

(v) (– 4, 2, 5) (vi) (–3, –1, 6) (vii) (2, – 4, – 7) (viii) (– 4, 2, – 5).

3. Let A, B, C be the feet of perpendiculars from a point P on the x, y, z-axis respectively. Find the coordinates of A, B and C in each of the following where the point P is :

(i) A = (3, 4, 2) (ii) (–5, 3, 7) (iii) (4, – 3, – 5)

4. Let A, B, C be the feet of perpendiculars from a point P on the *xy*, *yz* and *zx*-planes respectively. Find the coordinates of A, B, C in each of the following where the point P is

(i) (3, 4, 5) (ii) (–5, 3, 7) (iii) (4, – 3, – 5).

5. How far apart are the points (2, 0, 0) and (–3, 0, 0)?

6. Find the distance from the origin to (6, 6, 7).

7. Show that if $x^2 + y^2 = 1$, then the point $(x, y, \sqrt{1-x^2-y^2})$ is at a distance 1 unit from the origin.

8. Show that the point A (1, – 1, 3), B (2, – 4, 5) and (5, – 13, 11) are collinear.

9. Three consecutive vertices of a parallelogram ABCD are A (6, – 2, 4), B (2, 4, – 8), C (–2, 2, 4). Find the coordinates of the fourth vertex.

[**Hint:** Diagonals of a parallelogram have the same mid-point.]

10. Show that the triangle ABC with vertices A (0, 4, 1), B (2, 3, – 1) and C (4, 5, 0) is right angled.

11. Find the third vertex of triangle whose centroid is origin and two vertices are (2, 4, 6) and (0, –2, –5).

12. Find the centroid of a triangle, the mid-point of whose sides are D (1, 2, – 3), E (3, 0, 1) and F (– 1, 1, – 4).

13. The mid-points of the sides of a triangle are (5, 7, 11), (0, 8, 5) and (2, 3, – 1). Find its vertices.

14. Three vertices of a Parallelogram ABCD are A (1, 2, 3), B (– 1, – 2, – 1) and C (2, 3, 2). Find the fourth vertex D.

15. Find the coordinate of the points which trisect the line segment joining the points A (2, 1, – 3) and B (5, – 8, 3).

16. If the origin is the centriod of a triangle ABC having vertices A (*a*, 1, 3), B (– 2, *b*, – 5) and C (4, 7, *c*), find the values of *a*, *b*, *c*.

17. Let A (2, 2, – 3), B (5, 6, 9) and C (2, 7, 9) be the vertices of a triangle. The internal bisector of the angle A meets BC at the point D. Find the coordinates of D.

Long Answer Type

18. Show that the three points A (2, 3, 4), B (–1, 2, – 3) and C (– 4, 1, – 10) are collinear and find the ratio in which C divides AB.

19. The mid-point of the sides of a triangle are (1, 5, – 1), (0, 4, – 2) and (2, 3, 4). Find its vertices. Also find the centriod of the triangle.

20. Prove that the points (0, – 1, – 7), (2, 1, – 9) and (6, 5, – 13) are collinear. Find the ratio in which the first point divides the join of the other two.

21. What are the coordinates of the vertices of a cube whose edge is 2 units, one of whose vertices coincides with the origin and the three edges passing through the origin, coincides with the positive direction of the axes through the origin?

Objective Type Questions

Choose the correct answer from the given four options inidcated against each of the Exercises from 22 (M.C.Q.).

22. The distance of point P(3, 4, 5) from the *yz*-plane is

(A) 3 units (B) 4 units (C) 5 units (D) 550

23. What is the length of foot of perpendicular drawn from the point P (3, 4, 5) on *y*-axis

(A) $\sqrt{41}$ (B) $\sqrt{34}$ (C) 5 (D) none of these

24. Distance of the point (3, 4, 5) from the origin (0, 0, 0) is

(A) $\sqrt{50}$ (B) 3 (C) 4 (D) 5

25. If the distance between the points (*a*, 0, 1) and (0, 1, 2) is $\sqrt{27}$, then the value of *a* is

(A) 5 (B) ± 5 (C) – 5 (D) none of these

26. *x*-axis is the intersection of two planes

(A) *xy* and *xz* (B) *yz* and *zx* (C) *xy* and *yz* (D) none of these

27. Equation of *y*-axis is considered as

(A) $x = 0, y = 0$ (B) $y = 0, z = 0$ (C) $z = 0, x = 0$ (D) none of these

28. The point (–2, –3, –4) lies in the

(A) First octant (B) Seventh octant

(C) Second octant (D) Eighth octant

29. A plane is parallel to *yz*-plane so it is perpendicular to :

(A) *x*-axis (B) *y*-axis (C) *z*-axis (D) none of these

30. The locus of a point for which $y = 0, z = 0$ is

(A) equation of *x*-axis (B) equation of *y*-axis

(C) equation at *z*-axis (D) none of these

31. The locus of a point for which $x = 0$ is

(A) xy-plane (B) yz-plane (C) zx-plane (D) none of these

32. If a parallelopiped is formed by planes drawn through the points (5, 8, 10) and (3, 6, 8) parallel to the coordinate planes, then the length of diagonal of the parallelopiped is

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

33. L is the foot of the perpendicular drawn from a point P (3, 4, 5) on the xy-plane. The coordinates of point L are

(A) (3, 0, 0) (B) (0, 4, 5) (C) (3, 0, 5) (D) none of these

34. L is the foot of the perpendicular drawn from a point (3, 4, 5) on x-axis. The coordinates of L are

(A) (3, 0, 0) (B) (0, 4, 0) (C) (0, 0, 5) (D) none of these

Fill in the blanks in Exercises from 35 to 49.

35. The three axes OX, OY, OZ determine ________ .

36. The three planes determine a rectangular parallelopiped which has ________ of rectangular faces.

37. The coordinates of a point are the perpendicular distance from the ________ on the respectives axes.

38. The three coordinate planes divide the space into ________ parts.

39. If a point P lies in yz-plane, then the coordinates of a point on yz-plane is of the form ________.

40. The equation of yz-plane is ________.

41. If the point P lies on z-axis, then coordinates of P are of the form ________.

42. The equation of z-axis, are ________.

43. A line is parallel to xy-plane if all the points on the line have equal ________.

44. A line is parallel to x-axis if all the points on the line have equal ________.

45. $x = a$ represent a plane parallel to ________.

46. The plane parallel to yz - plane is perpendicular to ________.

47. The length of the longest piece of a string that can be stretched straight in a rectangular room whose dimensions are 10, 13 and 8 units are ______.

48. If the distance between the points $(a, 2, 1)$ and $(1, -1, 1)$ is 5, then a ________.

49. If the mid-points of the sides of a triangle AB; BC; CA are D $(1, 2, -3)$, E $(3, 0, 1)$ and F $(-1, 1, -4)$, then the centriod of the triangle ABC is ________.

50. Match each item given under the column C_1 to its correct answer given under column C_2.

	Column C_1		**Column C_2**
(b)	Point (2, 3,4) lies in the	(ii)	*yz*-plane
(c)	Locus of the points having x coordinate 0 is	(iii)	*z*-coordinate is zero
(d)	A line is parallel to *x*-axis if and only	(iv)	*z*-axis
(e)	If $x = 0$, $y = 0$ taken together will represent the	(v)	plane parallel to *xy*-plane
(f)	$z = c$ represent the plane	(vi)	if all the points on the line have equal y and *z*-coordinates.
(g)	Planes $x = a$, $y = b$ represent the line	(vii)	from the point on the respective
(h)	Coordinates of a point are the distances from the origin to the feet of perpendiculars	(viii)	parallel to z - axis.
(i)	A ball is the solid region in the space enclosed by a	(ix)	disc
(j)	Region in the plane enclosed by a circle is known as a	(x)	sphere

Chapter 13

LIMITS AND DERIVATIVES

13.1 Overview

13.1.1 *Limits of a function*

Let f be a function defined in a domain which we take to be an interval, say, I. We shall study the concept of limit of f at a point 'a' in I.

We say $\lim_{x \to a^-} f(x)$ is the expected value of f at $x = a$ given the values of f near to the left of a. This value is called the *left hand limit* of f at a.

We say $\lim_{x \to a^+} f(x)$ is the expected value of f at $x = a$ given the values of f near to the right of a. This value is called the *right hand limit* of f at a.

If the right and left hand limits coincide, we call the common value as the limit of f at $x = a$ and denote it by $\lim_{x \to a} f(x)$.

Some properties of limits

Let f and g be two functions such that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

(i) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(iii) For every real number α

$$\lim_{x \to a} (\alpha f)(x) = \alpha \lim_{x \to a} f(x)$$

(iv) $\lim_{x \to a} [f(x)\, g(x)] = [\lim_{x \to a} f(x) \lim_{x \to a} g(x)]$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ provided } g(x) \neq 0$$

Limits of polynomials and rational functions

If f is a polynomial function, then $\lim_{x \to a} f(x)$ exists and is given by

$$\lim_{x \to a} f(x) = f(a)$$

An Important limit

An important limit which is very useful and used in the sequel is given below:

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Remark The above expression remains valid for any rational number provided 'a' is positive.

Limits of trigonometric functions

To evaluate the limits of trigonometric functions, we shall make use of the following limits which are given below:

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (ii) $\lim_{x \to 0} \cos x = 1$ (iii) $\lim_{x \to 0} \sin x = 0$

13.1.2 *Derivatives* Suppose f is a real valued function, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{... (1)}$$

is called the **derivative** of f at x, provided the limit on the R.H.S. of (1) exists.

Algebra of derivative of functions Since the very definition of derivatives involve limits in a rather direct fashion, we expect the rules of derivatives to follow closely that of limits as given below:

Let f and g be two functions such that their derivatives are defined in a common domain. Then :

(i) Derivative of the sum of two function is the sum of the derivatives of the functions.

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

(ii) Derivative of the difference of two functions is the difference of the derivatives of the functions.

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

(iii) Derivative of the product of two functions is given by the following *product rule*.

$$\frac{d}{dx}[f(x)\cdot g(x)] = \left(\frac{d}{dx}f(x)\right)\cdot g(x) + f(x)\cdot\left(\frac{d}{dx}g(x)\right)$$

This is referred to as Leibnitz Rule for the product of two functions.

(iv) Derivative of quotient of two functions is given by the following *quotient rule* (wherever the denominator is non-zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{d}{dx}f(x)\right)\cdot g(x) - f(x)\cdot\left(\frac{d}{dx}g(x)\right)}{(g(x))^2}$$

13.2 Solved Examples

Short Answer Type

Example 1 Evaluate $\lim\limits_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right]$

Solution We have

$$\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right] = \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2}\right]$$

$$= \lim_{x\to 2}\left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right] \quad [x-2 \neq 0]$$

$$= \lim_{x\to 2}\left[\frac{x-3}{x(x-1)}\right] = \frac{-1}{2}$$

Example 2 Evaluate $\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$

Solution Put $y = 2 + x$ so that when $x \to 0$, $y \to 2$. Then

$$\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{y \to 2} \frac{y^{\frac{1}{2}} - 2^{\frac{1}{2}}}{y - 2}$$

$$= \frac{1}{2}(2)^{\frac{1}{2}-1} = \frac{1}{2} \cdot 2^{-\frac{1}{2}} = \frac{1}{2\sqrt{2}}$$

Example 3 Find the positive integer n so that $\lim_{x \to 3} \frac{x^n - 3^n}{x - 3} = 108$.

Solution We have

$$\lim_{x \to 3} \frac{x^n - 3^n}{x - 3} = n(3)^{n-1}$$

Therefore, $n(3)^{n-1} = 108 = 4\,(27) = 4(3)^{4-1}$

Comparing, we get $n = 4$

Example 4 Evaluate $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

Solution Put $y = \frac{\pi}{2} - x$. Then $y \to 0$ as $x \to \frac{\pi}{2}$. Therefore

$$\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{y \to 0} [\sec(\frac{\pi}{2} - y) - \tan(\frac{\pi}{2} - y)]$$

$$= \lim_{y \to 0} (\operatorname{cosec} y - \cot y)$$

$$= \lim_{y \to 0} \left(\frac{1}{\sin y} - \frac{\cos y}{\sin y} \right)$$

$$= \lim_{y \to 0} \left(\frac{1 - \cos y}{\sin y} \right)$$

$$= \lim_{y\to 0}\frac{2\sin^2\frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}} \qquad \left(\begin{array}{l}\text{since}, \sin^2\frac{y}{2}=\frac{1-\cos y}{2}\\ \sin y = 2\sin\frac{y}{2}\cos\frac{y}{2}\end{array}\right)$$

$$= \lim_{\frac{y}{2}\to 0}\tan\frac{y}{2} = 0$$

Example 5 Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x}$

Solution (i) We have

$$\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x} = \lim_{x\to 0}\frac{2\cos\frac{(2+x+2-x)}{2}\sin\frac{(2+x-2+x)}{2}}{x}$$

$$= \lim_{x\to 0}\frac{2\cos 2\sin x}{x}$$

$$= 2\cos 2\lim_{x\to 0}\frac{\sin x}{x} = 2\cos 2 \quad \left(\text{as} \lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

Example 6 Find the derivative of $f(x) = ax + b$, where a and b are non-zero constants, by first principle.

Solution By definition,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{a(x+h)+b-(ax+b)}{h} = \lim_{h\to 0}\frac{bh}{h} = b$$

Example 7 Find the derivative of $f(x) = ax^2 + bx + c$, where a, b and c are none-zero constant, by first principle.

Solution By definition,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h)^2 + b(x+h) + c - ax^2 - bx - c}{h}$$

$$= \lim_{h \to 0} \frac{bh + ah^2 + 2axh}{h} = \lim_{h \to 0} ah + 2ax + b = b + 2ax$$

Example 8 Find the derivative of $f(x) = x^3$, by first principle.

Solution By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + h^3 + 3xh(x+h) - x^3}{h}$$

$$= \lim_{h \to 0} (h^2 + 3x(x+h)) = 3x^2$$

Example 9 Find the derivative of $f(x) = \frac{1}{x}$ by first principle.

Solution By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$$

$$= \lim_{h \to 0} \frac{-h}{h(x+h)x} = \frac{-1}{x^2}.$$

Example 10 Find the derivative of $f(x) = \sin x$, by first principle.

Solution By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{2 \cdot \frac{h}{2}}$$

$$= \lim_{h \to 0} \cos\frac{(2x+h)}{2} \cdot \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= \cos x . 1 = \cos x$$

Example 11 Find the derivative of $f(x) = x^n$, where n is positive integer, by first principle.

Solution By definition,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{(x+h)^n - x^n}{h}$$

Using Binomial theorem, we have $(x + h)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} h + ... + {}^nC_n\, h^n$

Thus, $$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{h\,(nx^{n-1} + ... + h^{n-1}]}{h} = nx^{n-1}.$$

Example 12 Find the derivative of $2x^4 + x$.

Solution Let $y = 2x^4 + x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2x^4) + \frac{d}{dx}(x)$$

$$= 2 \times 4x^{4-1} + 1x^0$$

$$= 8x^3 + 1$$

Therefore, $$\frac{d}{dx}(2x^4 + x) = 8x^3 + 1.$$

Example 13 Find the derivative of x^2 cos$x$.

Solution Let $y = x^2 \cos x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cos x)$$

$$= x^2 \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2)$$

$$= x^2 (-\sin x) + \cos x\ (2x)$$

$$= 2x \cos x - x^2 \sin x$$

Long Answer Type

Example 14 Evaluate $\lim\limits_{x \to \frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$

Solution Note that

$$2\sin^2 x + \sin x - 1 = (2\sin x - 1)(\sin x + 1)$$

$$2\sin^2 x - 3\sin x + 1 = (2\sin x - 1)(\sin x - 1)$$

Therefore, $$\lim_{x \to \frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1} = \lim_{x \to \frac{\pi}{6}} \frac{(2\sin x - 1)(\sin x + 1)}{(2\sin x - 1)(\sin x - 1)}$$

$$= \lim_{x \to \frac{\pi}{6}} \frac{\sin x + 1}{\sin x - 1} \qquad (\text{as } 2\sin x - 1 \neq 0)$$

$$= \frac{1 + \sin\frac{\pi}{6}}{\sin\frac{\pi}{6} - 1} = -3$$

Example 15 Evaluate $\lim_{x \to 0} \dfrac{\tan x - \sin x}{\sin^3 x}$

Solution We have

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x} = \lim_{x \to 0} \frac{\sin x \left(\dfrac{1}{\cos x} - 1\right)}{\sin^3 x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\cos x \sin^2 x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 \dfrac{x}{2}}{\cos x \left(4\sin^2 \dfrac{x}{2} \cdot \cos^2 \dfrac{x}{2}\right)} = \frac{1}{2}.$$

Example 16 Evaluate $\lim_{x \to a} \dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$

Solution We have $\lim_{x \to a} \dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$

$$= \lim_{x \to a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}}$$

$$= \lim_{x \to a} \frac{a + 2x - 3x}{\left(\sqrt{3a+x} - 2\sqrt{x}\right)\left(\sqrt{a+2x} + \sqrt{3x}\right)}$$

$$= \lim_{x \to a} \frac{(a-x)\left(\sqrt{3a+x} + 2\sqrt{x}\right)}{\left(\sqrt{a+2x} + \sqrt{3x}\right)\left(\sqrt{3a+x} - 2\sqrt{x}\right)\left(\sqrt{3a+x} + 2\sqrt{x}\right)}$$

$$= \lim_{x \to a} \frac{(a-x)\left[\sqrt{3a+x} + 2\sqrt{x}\right]}{\left(\sqrt{a+2x} + \sqrt{3x}\right)(3a + x - 4x)}$$

$$= \frac{4\sqrt{a}}{3 \times 2\sqrt{3a}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

Example 17 Evaluate $\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$

Solution We have $\lim_{x \to 0} \frac{2\sin\left(\frac{(a+b)}{2}x\right)\sin\frac{(a-b)x}{2}}{2\sin^2\frac{cx}{2}}$

$$= \lim_{x \to 0} \frac{2\sin\frac{(a+b)x}{2} \cdot \sin\frac{(a-b)x}{2}}{x^2} \cdot \frac{x^2}{\sin^2\frac{cx}{2}}$$

$$= \lim_{x \to 0} \frac{\sin\frac{(a+b)x}{2}}{\frac{(a+b)x}{2} \cdot \left(\frac{2}{a+b}\right)} \cdot \frac{\sin\frac{(a-b)x}{2}}{\frac{(a-b)x}{2} \cdot \frac{2}{a-b}} \cdot \frac{\left(\frac{cx}{2}\right)^2 \times \frac{4}{c^2}}{\sin^2\frac{cx}{2}}$$

$$= \left(\frac{a+b}{2} \times \frac{a-b}{2} \times \frac{4}{c^2}\right) = \frac{a^2-b^2}{c^2}$$

Example 18 Evaluate $\lim_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

Solution We have $\lim_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

$$= \lim_{h \to 0} \frac{(a^2+h^2+2ah)[\sin a \cos h + \cos a \sin h] - a^2 \sin a}{h}$$

$$= \lim_{h \to 0} \left[\frac{a^2 \sin a(\cos h - 1)}{h} + \frac{a^2 \cos a \sin h}{h} + (h+2a)(\sin a \cos h + \cos a \sin h)\right]$$

$$= \lim_{h\to 0}\left[\frac{a^2 \sin a\,(-2\sin^2 \frac{h}{2})}{\frac{h^2}{2}}\cdot\frac{h}{2}\right] + \lim_{h\to 0}\frac{a^2 \cos a \sin h}{h} + \lim_{h\to 0}(h+2a)\sin(a+h)$$

$= a^2 \sin a \times 0 + a^2 \cos a\ (1) + 2a \sin a$

$= a^2 \cos a + 2a \sin a.$

Example 19 Find the derivative of $f(x) = \tan(ax + b)$, by first principle.

Solution We have $f'(x) = \lim_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0}\frac{\tan\big(a(x+h)+b\big) - \tan(ax+b)}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{\sin(ax+ah+b)}{\cos(ax+ah+b)} - \dfrac{\sin(ax+b)}{\cos(ax+b)}}{h}$$

$$= \lim_{h\to 0}\frac{\sin(ax+ah+b)\cos(ax+b) - \sin(ax+b)\cos(ax+ah+b)}{h\cos(ax+b)\cos(ax+ah+b)}$$

$$= \lim_{h\to 0}\frac{a\sin(ah)}{a\cdot h\cos(ax+b)\cos(ax+ah+b)}$$

$$= \lim_{h\to 0}\frac{a}{\cos(ax+b)\cos(ax+ah+b)}\ \lim_{ah\to 0}\frac{\sin ah}{ah}\quad [\text{as } h\to 0\ \ ah\to 0]$$

$$= \frac{a}{\cos^2(ax+b)} = a\sec^2(ax+b).$$

Example 20 Find the derivative of $f(x) = \sqrt{\sin x}$, by first principle.

Solution By definition,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{\sin(x+h)} - \sqrt{\sin x}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{\sin(x+h)} - \sqrt{\sin x}\right)\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}{h\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{2\cdot\frac{h}{2}\left(\sqrt{\sin(x+h)} + \sqrt{\sin x}\right)}$$

$$= \frac{\cos x}{2\sqrt{\sin x}} = \frac{1}{2}\cot x\sqrt{\sin x}$$

Example 21 Find the derivative of $\frac{\cos x}{1+\sin x}$.

Solution Let $y = \frac{\cos x}{1+\sin x}$

Differentiating both sides with respects to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \frac{(1+\sin x)\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$$

Objective Type Questions

Choose the correct answer out of the four options given against each Example 22 to 28 (M.C.Q.).

Example22 $\lim_{x\to 0} \frac{\sin x}{x(1+\cos x)}$ is equal to

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) –1

Solution (B) is the correct answer, we have

$$\lim_{x\to 0} \frac{\sin x}{x(1+\cos x)} = \lim_{x\to 0} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{x\left(2\cos^2\frac{x}{2}\right)}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{\tan\frac{x}{2}}{\frac{x}{2}} = \frac{1}{2}$$

Example23 $\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos x}$ is equal to

(A) 0 (B) –1 (C) 1 (D) does not exit

Solution (A) is the correct answer, since

$$\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos x} = \lim_{y\to 0}\left[\frac{1-\sin\left(\frac{\pi}{2}-y\right)}{\cos\left(\frac{\pi}{2}-y\right)}\right] \left(\text{taking } \frac{\pi}{2}-x=y\right)$$

$$= \lim_{y \to 0} \frac{1-\cos y}{\sin y} = \lim_{y \to 0} \frac{2\sin^2 \frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}}$$

$$= \lim_{y \to 0} \tan\frac{y}{2} = 0$$

Example 24 $\lim_{x \to 0} \frac{|x|}{x}$ is equal to

(A) 1 (B) –1 (C) 0 (D) does not exists

Solution (D) is the correct answer, since

$$\text{R.H.S} = \lim_{x \to 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

and $$\text{L.H.S} = \lim_{x \to 0^-} \frac{|x|}{x} = \frac{-x}{x} = -1$$

Example 25 $\lim_{x \to 1} [x-1]$, where [.] is greatest integer function, is equal to

(A) 1 (B) 2 (C) 0 (D) does not exists

Solution (D) is the correct answer, since

$$\text{R.H.S} = \lim_{x \to 1^+} [x-1] = 0$$

and $$\text{L.H.S} = \lim_{x \to 1^-} [x-1] = -1$$

Example 26 $\lim_{x \to 0} x\sin\frac{1}{x}$ is equals to

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) does not exist

Solution (A) is the correct answer, since

$\lim_{x \to 0} x = 0$ and $-1 \le \sin\frac{1}{x} \le 1$, by Sandwitch Theorem, we have

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$

Example 27 $\lim_{n \to \infty} \frac{1+2+3+...+n}{n^2}$, $n \in \mathbf{N}$, is equal to

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Solution (C) is the correct answer. As $\lim_{x \to \infty} \frac{1+2+3+...+n}{n^2}$

$$= \lim_{n \to \infty} \frac{n(n+1)}{2n^2} = \lim_{x \to \infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

Example 28 If $f(x) = x \sin x$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(A) 0 (B) 1 (C) –1 (D) $\frac{1}{2}$

Solution (B) is the correct answer. As $f'(x) = x \cos x + \sin x$

So, $$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$$

13.3 EXERCISE

Short Answer Type

Evaluate :

1. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$
2. $\lim_{x \to \frac{1}{2}} \frac{4x^2 - 1}{2x - 1}$
3. $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$
4. $\lim_{x \to 0} \frac{(x+2)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x}$
5. $\lim_{x \to 1} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$
6. $\lim_{x \to a} \frac{(2+x)^{\frac{5}{2}} - (a+2)^{\frac{5}{2}}}{x - a}$

7. $\lim_{x \to 1} \frac{x^4 - \sqrt{x}}{\sqrt{x} - 1}$

8. $\lim_{x \to 2} \frac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}}$

9. $\lim_{x \to \sqrt{2}} \frac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8}$

10. $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$

11. $\lim_{x \to 0} \frac{\sqrt{1 + x^3} - \sqrt{1 - x^3}}{x^2}$

12. $\lim_{x \to -3} \frac{x^3 + 27}{x^5 + 243}$

13. $\lim_{x \to \frac{1}{2}} \left(\frac{8x - 3}{2x - 1} - \frac{4x^2 + 1}{4x^2 - 1} \right)$

14. Find 'n', if $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$, $n \in \mathbf{N}$

15. $\lim_{x \to a} \frac{\sin 3x}{\sin 7x}$

16. $\lim_{x \to 0} \frac{\sin^2 2x}{\sin^2 4x}$

17. $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$

18. $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$

19. $\lim_{x \to 0} \frac{1 - \cos mx}{1 - \cos nx}$

20. $\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)}$

21. $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$

22. $\lim_{x \to \frac{\pi}{6}} \frac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}}$

23. $\lim_{x \to 0} \frac{\sin 2x + 3x}{2x + \tan 3x}$

24. $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$

25. $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

26. $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$

27. $\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$

28. If $\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$, then find the value of k.

Differentiate each of the functions w. r. to x in Exercises 29 to 42.

29. $\frac{x^4 + x^3 + x^2 + 1}{x}$

30. $\left(x + \frac{1}{x} \right)^3$

31. $(3x + 5)(1 + \tan x)$

32. $(\sec x - 1)(\sec x + 1)$ **33.** $\dfrac{3x+4}{5x^2-7x+9}$ **34.** $\dfrac{x^5-\cos x}{\sin x}$

35. $\dfrac{x^2\cos\frac{\pi}{4}}{\sin x}$ **36.** $(ax^2 + \cot x)(p + q\cos x)$

37. $\dfrac{a+b\sin x}{c+d\cos x}$ **38.** $(\sin x + \cos x)^2$ **39.** $(2x - 7)^2 (3x + 5)^3$

40. $x^2 \sin x + \cos 2x$ **41.** $\sin^3 x \cos^3 x$ **42.** $\dfrac{1}{ax^2+bx+c}$

Long Answer Type

Differentiate each of the functions with respect to 'x' in Exercises 43 to 46 using first principle.

43. $\cos(x^2 + 1)$ **44.** $\dfrac{ax+b}{cx+d}$ **45.** $x^{\frac{2}{3}}$

46. $x\cos x$

Evaluate each of the following limits in Exercises 47 to 53.

47. $\lim\limits_{y\to 0}\dfrac{(x+y)\sec(x+y)-x\sec x}{y}$

48. $\lim\limits_{x\to 0}\dfrac{(\sin(\alpha+\beta)x+\sin(\alpha-\beta)x+\sin 2\alpha x)}{\cos 2\beta x-\cos 2\alpha x}\cdot x$

49. $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}$ **50.** $\lim\limits_{x\to\pi}\dfrac{1-\sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}$

51. Show that $\lim\limits_{x\to 4}\dfrac{|x-4|}{x-4}$ does not exists

52. Let $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x} & \text{when } x \neq \dfrac{\pi}{2} \\ 3 & x = \dfrac{\pi}{2} \end{cases}$ and if $\lim\limits_{x \to \frac{\pi}{2}} f(x) = f(\frac{\pi}{2})$,

find the value of k.

53. Let $f(x) = \begin{cases} x+2 & x \leq -1 \\ cx^2 & x > -1 \end{cases}$, find '$c$' if $\lim\limits_{x \to -1} f(x)$ exists.

Objective Type Questions

Choose the correct answer out of 4 options given against each Exercise 54 to 76 (M.C.Q).

54. $\lim\limits_{x \to \pi} \dfrac{\sin x}{x - \pi}$ is

(A) 1 (B) 2 (C) –1 (D) –2

55. $\lim\limits_{x \to 0} \dfrac{x^2 \cos x}{1 - \cos x}$ is

(A) 2 (B) $\dfrac{3}{2}$ (C) $\dfrac{-3}{2}$ (D) 1

56. $\lim\limits_{x \to 0} \dfrac{(1+x)^n - 1}{x}$ is

(A) n (B) 1 (C) $-n$ (D) 0

57. $\lim\limits_{x \to 1} \dfrac{x^m - 1}{x^n - 1}$ is

(A) 1 (B) $\dfrac{m}{n}$ (C) $-\dfrac{m}{n}$ (D) $\dfrac{m^2}{n^2}$

58. $\lim\limits_{x \to 0} \dfrac{1 - \cos 4\theta}{1 - \cos 6\theta}$ is

(A) $\frac{4}{9}$ (B) $\frac{1}{2}$ (C) $\frac{-1}{2}$ (D) –1

59. $\lim_{x \to 0} \frac{\operatorname{cosec} x - \cot x}{x}$ is

(A) $\frac{-1}{2}$ (B) 1 (C) $\frac{1}{2}$ (D) 1

60. $\lim_{x \to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$ is

(A) 2 (B) 0 (C) 1 (D) –1

61. $\lim_{x \to \frac{\pi}{4}} \frac{\sec^2 x - 2}{\tan x - 1}$ is

(A) 3 (B) 1 (C) 0 (D) $\sqrt{2}$

62. $\lim_{x \to 1} \frac{\left(\sqrt{x} - 1\right)\left(2x - 3\right)}{2x^2 + x - 3}$ is

(A) $\frac{1}{10}$ (B) $\frac{-1}{10}$ (C) 1 (D) None of these

63. If $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where [.] denotes the greatest integer function ,

then $\lim_{x \to 0} f(x)$ is equal to

(A) 1 (B) 0 (C) –1 (D) None of these

64. $\lim_{x \to 0} \frac{|\sin x|}{x}$ is

(A) 1 (B) –1 (C) does not exist (D) None of these

65. Let $f(x) = \begin{cases} x^2 - 1, & 0 < x < 2 \\ 2x + 3, & 2 \leq x < 3 \end{cases}$, the quadratic equation whose roots are $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$ is

(A) $x^2 - 6x + 9 = 0$ (B) $x^2 - 7x + 8 = 0$

(C) $x^2 - 14x + 49 = 0$ (D) $x^2 - 10x + 21 = 0$

66. $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x}$ is

(A) 2 (B) $\frac{1}{2}$ (C) $\frac{-1}{2}$ (D) $\frac{1}{4}$

67. Let $f(x) = x - [x]; \in \mathbf{R}$, then $f'\left(\frac{1}{2}\right)$ is

(A) $\frac{3}{2}$ (B) 1 (C) 0 (D) –1

68. If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at $x = 1$ is

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) 0

69. If $f(x) = \frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is

(A) $\frac{5}{4}$ (B) $\frac{4}{5}$ (C) 1 (D) 0

70. If $y = \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$, then $\frac{dy}{dx}$ is

(A) $\frac{-4x}{(x^2-1)^2}$ (B) $\frac{-4x}{x^2-1}$ (C) $\frac{1-x^2}{4x}$ (D) $\frac{4x}{x^2-1}$

71. If $y = \frac{\sin x + \cos x}{\sin x - \cos x}$, then $\frac{dy}{dx}$ at $x = 0$ is

(A) –2 (B) 0 (C) $\frac{1}{2}$ (D) does not exist

72. If $y = \frac{\sin(x+9)}{\cos x}$ then $\frac{dy}{dx}$ at $x = 0$ is

(A) cos 9 (B) sin 9 (C) 0 (D) 1

73. If $f(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^{100}}{100}$, then $f'(1)$ is equal to

(A) $\frac{1}{100}$ (B) 100 (C) does not exist (D) 0

74. If $f(x) = \frac{x^n - a^n}{x - a}$ for some constant 'a', then $f'(a)$ is

(A) 1 (B) 0 (C) does not exist (D) $\frac{1}{2}$

75. If $f(x) = x^{100} + x^{99} + \ldots + x + 1$, then $f'(1)$ is equal to

(A) 5050 (B) 5049 (C) 5051 (D) 50051

76. If $f(x) = 1 - x + x^2 - x^3 \ldots - x^{99} + x^{100}$, then $f'(1)$ is euqal to

(A) 150 (B) –50 (C) –150 (D) 50

Fill in the blanks in Exercises 77 to 80.

77. If $f(x) = \frac{\tan x}{x - \pi}$, then $\lim_{x \to \pi} f(x) =$ ______________

78. $\lim_{x \to 0} \left(\sin mx \cot \frac{x}{\sqrt{3}} \right) = 2$, then $m =$ ______________

79. if $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, then $\frac{dy}{dx} =$ ______________

80. $\lim_{x \to 3^+} \frac{x}{[x]} =$ ______________

Chapter 14

MATHEMATICAL REASONING

14.1 Overview

If an object is either black or white, and if it is not black, then logic leads us to the conclusion that it must be white. Observe that logical reasoning from the given hypotheses can not reveal what "black" or "white" mean, or why an object can not be both. Infact, logic is the study of general patterns of reasoning, without reference to particular meaning or context.

14.1.1 *Statements*

A statement is a sentence which is either true or false, but not both simultaneously.

Note: No sentence can be called a statement if

(i) It is an exclamation

(ii) It is an order or request

(iii) It is a question

(iv) It involves variable time such as 'today', 'tomorrow', 'yesterday' etc.

(v) It involves variable places such as 'here', 'there', 'everywhere' etc.

(vi) It involves pronouns such as 'she', 'he', 'they' etc.

Example 1

(i) The sentence

'New Delhi is in India; is true. So it is a statement.

(ii) The sentence

"Every rectangle is a square" is false. So it is a statement.

(iii) The sentence

"Close the door" can not be assigned true or false (Infact, it is a command). So it can not be called a statement.

(iv) The sentence

"How old are you?" can not be assigned true or false (In fact, it is a question). So it is not a statement.

(v) The truth or falsity of the sentence

"x is a natural number" depends on the value of x. So it is not considered as a statement. However, in some books it is called an open statement.

Note: Truth and falisity of a statement is called its truth value.

14.1.2 ***Simple statements*** A statement is called simple if it can not be broken down into two or more statements.

Example 2 The statements

"2 is an even number",

"A square has all its sides equal" and

" Chandigarh is the capital of Haryana" are all simple statements.

14.1.3 ***Compound statements*** A compound statement is the one which is made up of two or more simple statements.

Example 3 The statement

"11 is both an odd and prime number" can be broken into two statements

"11 is an odd number" and "11 is a prime number" so it is a compound statement.

Note: The simple statements which constitutes a compound statement are called **component statements**.

14.1.4 ***Basic logical connectives*** There are many ways of combining simple statements to form new statements. The words which combine or change simple statements to form new statements or compound statements are called **Connectives**. The basic connectives (logical) *conjunction* corresponds to the English word 'and'; *disjunction* corresponds to the word 'or'; and *negation* corresponds to the word 'not'.

Throughout we use the symbol '$\wedge$' to denote conjunction; '$\vee$' to denote disjunction and the symbol '$\sim$' to denote negation.

Note: Negation is called a connective although it does not combine two or more statements. In fact, it only *modifies* a statement.

14.1.5 ***Conjunction*** If two simple statements p and q are connected by the word 'and', then the resulting compound statement "p and q" is called a conjunction of p and q and is written in symbolic form as "$p \wedge q$".

Example 4 Form the conjunction of the following simple statements:

p : Dinesh is a boy.

q : Nagma is a girl.

Solution The conjunction of the statement p and q is given by

$p \wedge q$: Dinesh is a boy and Nagma is a girl.

Example 5 Translate the following statement into symbolic form

"Jack and Jill went up the hill."

Solution The given statement can be rewritten as

"Jack went up the hill and Jill went up the hill"

Let p : Jack went up the hill and q : Jill went up the hill.

Then the given statement in symbolic form is $p \wedge q$.

Regarding the truth value of the conjunction $p \wedge q$ of two simple statements p and q, we have

(D_1) : *The statement $p \wedge q$ has the truth value T (true) whenever both p and q have the truth value T.*

(D_2) : *The statement $p \wedge q$ has the truth value F (false) whenever either p or q or both have the truth value F.*

Example 6 Write the truth value of each of the following four statements:

(i) Delhi is in India and 2 + 3 = 6.

(ii) Delhi is in India and 2 + 3 = 5.

(iii) Delhi is in Nepal and 2 + 3 = 5.

(iv) Delhi is in Nepal and 2 + 3 = 6.

Solution In view of (D_1) and (D_2) above, we observe that statement (i) has the truth value F as the truth value of the statement "2 + 3 = 6" is F. Also, statement (ii) has the truth value T as both the statement "Delhi is in India" and "2 + 3 = 5" has the truth value T.

Similarly, the truth value of both the statements (iii) and (iv) is F.

14.1.6 *Disjunction* If two simple statements p and q are connected by the word 'or', then the resulting compound statement "p or q" is called *disjunction* of p and q and is written in symbolic form as "$p \vee q$".

Example 7 Form the disjunction of the following simple statements:

p : The sun shines.

q : It rains.

Solution The disjunction of the statements p and q is given by

$p \vee q$: The sun shines or it rains.

Regarding the truth value of the disjunction $p \vee q$ of two simple statements p and q, we have

(D_3) : *The statement $p \vee q$ has the truth value F whenever both p and q have the truth value F.*

(D_4) : *The statement $p \vee q$ has the truth value T whenever either p or q or both have the truth value T.*

Example 8 Write the truth value of each of the following statements:

(i) India is in Asia or 2 + 2 = 4.

(ii) India is in Asia or 2 + 2 = 5.

(iii) India is in Europe or 2 + 2 = 4.

(iv) India is in Europe or 2 + 2 = 5.

Solution In view of (D_3) and (D_4) above, we observe that only the last statement has the truth value F as both the sub-statements "India is in Europe" and "2 + 2 = 5" have the truth value F. The remaining statements (i) to (iii) have the truth value T as at least one of the sub-statements of these statements has the truth value T.

14.1.7 *Negation* An assertion that a statement fails or denial of a statement is called the negation of the statement. The negation of a statement is generally formed by introducing the word "not" at some proper place in the statement or by prefixing the statement with "It is not the case that" or It is false that".

The negation of a statement p in symbolic form is written as "$\sim p$".

Example 9 Write the negation of the statement

p : New Delhi is a city.

Solution The negation of p is given by

$\sim p$: New Delhi is not a city

or $\sim p$: It is not the case that New Delhi is a city.

or $\sim p$: It is false that New Delhi is a city.

Regarding the truth value of the negation $\sim p$ of a statement p, we have

(D_5) : *$\sim p$ has truth value T whenever p has truth value F.*

(D_6) : *$\sim p$ has truth value F whenever p has truth value T.*

Example 10 Write the truth value of the negation of each of the following statements:

(i) p : Every square is a rectangle.

(ii) q : The earth is a star.

(iii) $r : 2 + 3 < 4$

Solution In view of (D_5) and (D_6), we observe that the truth value of $\sim p$ is F as the truth value of p is T. Similarly, the truth value of both $\sim q$ and $\sim r$ is T as the truth value of both statements q and r is F.

14.1.8 *Negation of compound statements*

14.1.9 ***Negation of conjunction*** Recall that a conjunction $p \wedge q$ consists of two component statements p and q both of which exist simultaneously. Therefore, the negation of the conjunction would mean the negation of at least one of the two component statements. Thus, we have

(D_7) : *The negation of a conjunction $p \wedge q$ is the disjunction of the negation of p and the negation of q. Equivalently, we write*

$$\sim (p \wedge q) = \sim p \vee \sim q$$

Example 11 Write the negation of each of the following conjunctions:

(a) Paris is in France and London is in England.

(b) $2 + 3 = 5$ and $8 < 10$.

Solution

(a) Write p : Paris is in France and q : London is in England.

Then, the conjunction in (a) is given by $p \wedge q$.

Now $\sim p$: Paris is not in France, and

$\sim q$: London is not in England.

Therefore, using (D_7), negation of $p \wedge q$ is given by

$\sim (p \wedge q)$ = Paris is not in France or London is not in England.

(b) Write $p : 2 + 3 = 5$ and $q : 8 < 10$.

Then the conjunction in (b) is given by $p \wedge q$.

Now $\sim p : 2 + 3 \neq 5$ and $\sim q : 8 \not< 10$.

Then, using (D_7), negation of $p \wedge q$ is given by

$– (p \wedge q) = (2 + 3 \neq 5)$ or $(8 \not< 10)$

14.1.10 ***Negation of disjunction*** Recall that a disjunction $p \vee q$ is consisting of two component statements p and q which are such that either p or q or both exist. Therefore, the negation of the disjunction would mean the negation of both p and q simultaneously. Thus, in symbolic form, we have

(D_8) : *The negation of a disjunction $p \vee q$ is the conjunction of the negation of p and the negation of q. Equivalently, we write*

$$\sim (p \vee q) = \sim p \wedge \sim q$$

Example 12 Write the negation of each of the following disjunction :

(a) Ram is in Class X or Rahim is in Class XII.

(b) 7 is greater than 4 or 6 is less than 7.

Solution

(a) Let p : Ram is in Class X and q : Rahim is in Class XII.

Then the disjunction in (a) is given by $p \vee q$.

Now $\sim p$: Ram is not in Class X.

$\sim q$: Rahim is not in Class XII.

Then, using (D_8), negation of $p \vee q$ is given by

$\sim (p \vee q)$: Ram is not in Class X and Rahim is not in Class XII.

(b) Write p : 7 is greater than 4, and q : 6 is less than 7.

Then, using (D_8), negation of $p \vee q$ is given by

$\sim (p \vee q)$: 7 is not greater than 4 and 6 is not less than 7.

14.1.11 ***Negation of a negation*** As already remarked the negation is not a connective but a modifier. It only modifies a given statement and applies only to a single simple statement. Therefore, in view of (D_5) and (D_6), for a statement p, we have

(D_9) : *Negation of negation of a statement is the statement itself. Equivalently, we write*

$$\sim (\sim p) = p$$

14.1.12 ***The conditional statement*** Recall that if p and q are any two statements, then the compound statement "**if p then q**" formed by joining p and q by a connective **'if then'** is called a *conditional statement* or an *implication* and is written in symbolic form as $p \rightarrow q$ or $p \Rightarrow q$. Here, p is called hypothesis (or antecedent) and q is called *conclusion* (or *consequent*) of the conditional statement ($p \Rightarrow q$):

Remark The conditional statement $p \Rightarrow q$ can be expressed in several different ways. Some of the common expressions are :

(a) if p, then q

(b) q if p

(c) p only if q

(d) p is sufficient for q

(e) q is necessary for p.

Observe that the conditional statement $p \rightarrow q$ reflects the idea that whenever it is known that p is true, it will have to follow that q is also true.

Example 13 Each of the following statements is also a conditional statement.

(i) If 2 + 2 = 5, then Rekha will get an ice-cream.

(ii) If you eat your dinner, then you will get dessert.

(iii) If John works hard, then it will rain today.

(iv) If ABC is a triangle, then $\angle A + \angle B + \angle C = 180°$.

Example 14 Express in English, the statement $p \rightarrow q$, where

p : it is raining today

q : $2 + 3 > 4$

Solution The required conditional statement is

"If it is raining today, then $2 + 3 > 4$*"*

14.1.13 *Contrapositive of a conditional statement* The statement "$(\sim q) \rightarrow (\sim p)$" is called the contrapositive of the statement $p \rightarrow q$

Example 15 Write each of the following statements in its equivalent contrapositive form:

(i) If my car is in the repair shop, then I cannot go to the market.

(ii) If Karim cannot swim to the *fort*, then he cannot swim across the river.

Solution (i) Let "p : my car is in the repair shop" and "q : I can not go to the market".

Then, the given statement in symbolic form is $p \rightarrow q$. Therefore, its contrapositive is given by $\sim q \rightarrow \sim p$.

Now $\sim p$: My car is not in the repair shop.

and $\sim q$: I can go to the market

Therefore, the contrapositive of the given statement is

"If I can go to the market, then my car is not in the repair shop".

(ii) Proceeding on the lines of the solution of (i), the contrapositive of the statement in (ii) is

"If Karim can swim across the river, then he can swim to the fort".

14.1.14 ***Converse of a conditional statement*** The conditional statement "$q \rightarrow p$" is called the converse of the conditional statement "$p \rightarrow q$"

Example 16 Write the converse of the following statements

(i) If $x < y$, then $x + 5 < y + 5$

(ii) If ABC is an equilateral triangle, then ABC is an isosceles triangle

Solution (i) Let

$p : x < y$

$q : x + 5 < y + 5$

Therefore, the converse of the statement $p \rightarrow q$ is given by

"If $x + 5 < y + 5$, then $x < y$

(ii) Converse of the given statement is

"If ABC is an isosceles triangle, then ABC is an equilateral triangle."

14.1.15 ***The biconditional statement*** If two statements p and q are connected by the connective 'if and only if' then the resulting compound statement "p if and only if q" is called a *biconditional* of p and q and is written in symbolic form as $p \leftrightarrow q$.

Example 17 Form the biconditional of the following statements:

p : One is less than seven

q : Two is less than eight

Solution The biconditional of p and q is given by "One is less than seven, if and only if two is less than eight".

Example 18 Translate the following biconditional into symbolic form:

"ABC is an equilateral triangle if and only if it is equiangular".

Solution Let p : ABC is an equilateral triangle

and q **:** ABC is an equiangular triangle.

Then, the given statement in symbolic form is given by $p \leftrightarrow q$.

14.1.16 ***Quantifiers*** Quantifieres are the phrases like 'These exist' and "for every". We come across many mathematical statement containing these phrases. For example – Consider the following statements

p : For every prime number x, $\sqrt{x}$ is an irrational number.

q : There exists a triangle whose all sides are equal.

14.1.17 ***Validity of statements*** Validity of a statement means checking when the statement is true and when it is not true. This depends upon which of the connectives, quantifiers and implication is being used in the statement.

(i) Validity of statement with 'AND'

To show statement $r : p \wedge q$ is true, show statement 'p' is true and statement 'q' is true.

(ii) Validity of statement with 'OR'

To show statement $r : p \vee q$ is true, show either statement 'p' is true or statement 'q' is true.

(iii) Validity of statement with "If-then"

To show statement r : "If p then q is true", we can adopt the following methods:

(a) Direct method : Assume p is true and show q is true, i.e., $p \Rightarrow q$.

(b) Contrapositive method : Assume $\sim q$ is true and show $\sim p$ is true, i.e., $\sim q \Rightarrow \sim p$.

(c) Contradiction method : Assume that p is true and q is false and obtain a contradiction from assumption.

(d) By giving a counter example : To prove the given statement r is false we give a counter example. Consider the follwoing statement.

"r : All prime numbers are odd". Now the statement 'r' is false as 2 is a prime number and it is an even number.

14.1.18 Validity of the statement with "If and only If" To show the statement $r : p$ if and only if q is true, we proceed as follows:

Step 1 Show if p is true then q is true.

Step 2 Show if q is true then p is true.

14.2 Solved Examples

Short Answer Type

Example 1 Which of the following statements are compound statements

(i) "2 is both an even number and a prime number"

(ii) "9 is neither an even number nor a prime number"

(iii) "Ram and Rahim are friends"

Solution

(i) The given statement can be broken into two simple statements "2 is an even number" and "2 is a prime number" and connected by the connective 'and'

(ii) The given statement can be broken into two simple statements

"9 is not an even number" and "9 is not a prime number" and connected by the connective 'and'

(iii) The given statement can not be broken into two simple statements and hence it is not a compound statement.

Example 2 Identify the component statements and the connective in the following compound statements.

(a) It is raining or the sun is shining.

(b) 2 is a positive number or a negative number.

Solution

(a) The component statements are given by

p : It is raining

q : The sun is shining

The connective is "or"

(b) The component statements are given by

p : 2 is a positive number

q : 2 is a negative number

The connective is 'or'

Example 3 Translate the following statements in symbolic form

(i) 2 and 3 are prime numbers

(ii) Tigers are found in Gir forest or Rajaji national park.

Solution

(i) The given statement can be rewritten as "2 is a prime number and 3 is a prime number".

Let p : 2 is a prime number

q : 3 is a prime number

Then the given statement in symbolic form is $p \wedge q$.

(ii) The given statement can be rewritten as

"Tigers are found in Gir forest or Tigers are found in Rajaji national park"

Let p : Tigers are found in Gir forest

q : Tigers are found in Rajaji national park.

Then the given statement in symbolic form is $p \vee q$.

Example 4 Write the truth value of each of the following statements.

(i) 9 is an even integer or 9 + 1 is even.

(ii) 2 + 4 = 6 or 2 + 4 = 7

(iii) Delhi is the capital of India and Islamabad is the capital of Pakistan.

(iv) Every rectangle is a square and every square is a rectangle.

(v) The sun is a star or sun is a planet.

Solution In view of (D_1), (D_2), (D_3) and (D_4), we observe that only statement (iv) has truth value F as the first component statement namely "every rectangle is a square" is false.

Further, in statements (i), (ii) and (v) atleast one component statement is true. Therefore, these statements have truth value T.

Also, truth value of statement (iii) is T as both the component statements are true.

Example 5 Write negation of the statement

"Everyone who lives in India is an Indian"

Solution Let p : Everyone who lives in India is an Indian. The negation of this statement is given by

$\sim p$: It is false that everyone who lives in India is an Indian.

or

$\sim p$: Everyone who lives in India is not an Indian.

Example 6 Write the negation of the following statements :

(a) p : All triangles are equilateral triangles.

(b) q : 9 is a multiple of 4.

(c) r : A triangle has four sides.

Solution

(a) We have

It is false that all triangles are equilateral triangles

or

$\sim p$: Threre exists a triangle which is not an equilateral triangles.

or

$\sim p$: Not all triangles are equilateral triangles

(b) $\sim q$: 9 is not a multiple of 4.

(c) $\sim r$: It is false that the triangle has four sides.

or

$\sim r$: A triangle has not four sides.

Example 7 Write the negation of the following statements :

(i) Suresh lives in Bhopal or he lives in Mumbai.

(ii) $x + y = y + x$ and 29 is a prime number.

Solution

(i) Let

p : Suresh lives in Bhopal

and q : Suresh lives in Mumbai

Then the disjunction in (i) is given by $p \vee q$.

Now $\sim p$: Suresh does not live in Bhopal.

$\sim q$: Suresh does not live in Mumbai.

Therefore, using (D_8), negation of $p \vee q$ is given by

$\sim (p \vee q)$: Suresh does not live in Bhopal and he does not live in Mumbai.

(ii) Let $p : x + y = y + x$

and q : 29 is a prime number.

Then the conjunction in (ii) is given by $p \wedge q$.

Now $\sim p : x + y \neq y + x$

and $\sim q$: 29 is not a prime number.

Therefore, using (D_7), negation of $p \wedge q$ is given by,

$\sim (p \wedge q) : x + y \neq y + x$ or 29 is not a prime number.

Example 8 Rewrite each of the following statements in the form of conditional statements :

(i) Mohan will be a good student if he studies hard.

(ii) Ramesh will get dessert only if he eats his dinner.

(iii) When you sing, my ears hurt.

(iv) A necessary condition for Indian team to win a cricket match is that the selection committee selects an all-rounder.

(v) A sufficient condition for Tara to visit New Delhi is that she goes to the Rashtrapati Bhawan.

Solution

(i) The given statement is of the form "q if p", where

p : Mohan studies hard.

q : He will be a good student.

It is an equivalent form (Remark (b) 14.1.12) of the statement "if p then q". So the equivalent formulation of the given statement is

"If Mohan studies hard, then he will be a good student".

(Here, note that in p he is replaced by Mohan and in q Mohan is replaced by he)

(ii) The given statement is of the form

"p only if q" which is an equivalent form (Remark (c) 14.1.12) of the statement "if p then q". So, the equivalent formulation of the given statement is:

"If Ramesh eats his dinner, then he will get dessert"

(iii) Here 'when' means the same as 'if ' and so the equivalent formulation of the given statements is:

"If you sing, then my ears hurt"

(iv) The given statement is of the form "q is necessary for p" where

p : Indian team wins a cricket match

q : The selection committee selects an all-rounder

which is an equivalent form (Remark (e) 14.1.12) of "if p then q". So the equivalent formulation of the given statement is

"If the teams wins a cricket match then selection committee selects an all rounder.

(v) The given statement is of the form "p is sufficient for q" where

p : Tara goes to Rashtrapati Bhawan

q : She visits New Delhi

which is an equivalent form (Remark (d) 14.1.12) of "if p, then q", so the equivalent formulation of the given statement is

"If Tara goes to Rashtrapati Bhawan, then she visits New Delhi".

Example 9 Express in English, the statement $p \rightarrow q$, where

p : It is raining today

q : $2 + 3 > 4$

Solution The conditional statement is

"If it is raining today, then $2 + 3 > 4$".

Example 10 Translate the following statements in symbolic form:

If $x = 7$ and $y = 4$" then $x + y = 11$.

Solution Let $p : x = 7$ and $y = 4$ and $q : x + y = 11$

Then the given statement is symbolic form is $p \rightarrow q$

Example 11 Form the biconditional of the following statements :

p : Today is 14th of August

q : Tomorrow is Independence day

Solution The biconditional $p \leftrightarrow q$ is given by

"Today is 14th of August if and only if tomorrow is Independence Day".

Example 12 Translate the following biconditional into symbolic form:

"ABC is an equilateral triangle if and only if its each interior angle is 60°"

Solution Let p : ABC is an equilateral triangle

and q : Each interior angle of triangle ABC is 60°

Then the given statement in symbolic form is $p \leftrightarrow q$.

Example 13 Identify the quantifiers and write the negation of the following statements

(i) There exists a number which is equal to its square.

(ii) For all even integers x, x^2 is also even.

(iii) There exists a number which is a multiple of 6 and 9.

Solution (i) The quantifier is "there exists" and the negation is

"There does not exist a number which is equal to its square"

(ii) The quantifier is "for all" and the negation is

"There exists an even integer x such that x^2 is not even"

(iii) The quantifier is "there exists" and the negation is

"There does not exist a number which is a multiple of both 6 and 9".

Example 14 Show that the following statement is true.

p : For any real numbers x, y if $x = y$, then $2x + a = 2y + a$ when $a \in$ **Z**.

Solution We prove the statement 'p' is true by contrapositive method and by Direct Method.

Direct Method for any real number x, y given

$x = y$

$\Rightarrow \quad 2x = 2y$

$\Rightarrow \quad 2x + a = 2y + a$ for some $a \in$ **Z**.

Contrapositve Method The contrapositive statement of 'p' is "For any real numbers x, y if $2x + a \neq 2y + a$, where $a \in$ **Z**, then $x \neq y$.

Given $\quad 2x + a \neq 2y + a$

$\Rightarrow \quad 2x \neq 2y$

$\Rightarrow \quad x \neq y$

Example 15 Check the validity of the statements

(i) r : 100 is a multiple of 4 and 5.

(ii) s : 60 is a multiple of 3 or 5.

Solution (i) Let $p : r \wedge s$

where $\quad r$: "100 is a multiple of 4" is true

s : "100 is a multiple of 5" is true

Hence p is true.

(ii) Let $q : r \vee s$, where

r : "60 is a multiple of 3", is true.

s : "60 is a multiple of 5", is true.

Hence q is true.

Objective Type Questions

Choose the correct answer out of the four options given against each of the Examples 16 to 18 (M.C.Q.).

Example 16 Which of the following is a statement?

(A) Roses are black.

(B) Mind your own business.

(C) Be punctual.

(D) Do not tell lies.

Solution (A) is the correct answer as the sentences in (B), (C) and (D) are neither true nor false. Infact all these sentences are advices.

Example 17 The negation of the statement

"It is raining and weather is cold." is

(A) It is not raining and weather is cold.

(B) It is raining or weather is not cold.

(C) It is not raining or weather is not cold.

(D) It is not raining and weather is not cold.

Solution (C) is the correct answer as it satisfies (D_7). The options (A), (B) and (D) do not satisfy (D_7).

Example 18 Which of the following is the converse of the statement?

"If Billu secure good marks, then he will get a bicycle."

(A) If Billu will not get bicycle, then he will not secure good marks.

(B) If Billu will get a bicycle, then he will secure good marks.

(C) If Billu will get a bicycle, then he will not secure good marks.

(D) If Billu will not get a bicycle, then he will secure good marks.

Solution (B) is the correct answer since the statement $q \rightarrow p$ is the converse of the statement $p \rightarrow q$.

14.3 EXERCISE

Short Answer Type

1. Which of the following sentences are statements? Justify
   (i) A triangle has three sides.
   (ii) 0 is a complex number.
   (iii) Sky is red.
   (iv) Every set is an infinite set.
   (v) $15 + 8 > 23$.
   (vi) $y + 9 = 7$.
   (vii) Where is your bag?
   (viii) Every square is a rectangle.
   (ix) Sum of opposite angles of a cyclic quadrilateral is 180°.
   (x) $\sin^2 x + \cos^2 x = 0$

2. Find the component statements of the following compound statements.

(i) Number 7 is prime and odd.

(ii) Chennai is in India and is the capital of Tamil Nadu.

(iii) The number 100 is divisible by 3, 11 and 5.

(iv) Chandigarh is the capital of Haryana and U.P.

(v) $\sqrt{7}$ is a rational number or an irrational number.

(vi) 0 is less than every positive integer and every negative integer.

(vii) Plants use sunlight, water and carbon dioxide for photosynthesis.

(viii) Two lines in a plane either intersect at one point or they are parallel.

(ix) A rectangle is a quadrilateral or a 5 - sided polygon.

3. Write the component statements of the following compound statements and check whether the compound statement is true or false.

(i) 57 is divisible by 2 or 3.

(ii) 24 is a multiple of 4 and 6.

(iii) All living things have two eyes and two legs.

(iv) 2 is an even number and a prime number.

4. Write the negation of the following simple statements

(i) The number 17 is prime.

(ii) $2 + 7 = 6$.

(iii) Violets are blue.

(iv) $\sqrt{5}$ is a rational number.

(v) 2 is not a prime number.

(vi) Every real number is an irrational number.

(vii) Cow has four legs.

(viii) A leap year has 366 days.

(ix) All similar triangles are congruent.

(x) Area of a circle is same as the perimeter of the circle.

5. Translate the following statements into symbolic form

(i) Rahul passed in Hindi and English.

(ii) x and y are even integers.

(iii) 2, 3 and 6 are factors of 12.

(iv) Either x or $x + 1$ is an odd integer.

(v) A number is either divisible by 2 or 3.

(vi) Either $x = 2$ or $x = 3$ is a root of $3x^2 - x - 10 = 0$

(vii) Students can take Hindi or English as an optional paper.

6. Write down the negation of following compound statements

(i) All rational numbers are real and complex.

(ii) All real numbers are rationals or irrationals.

(iii) $x = 2$ and $x = 3$ are roots of the Quadratic equation $x^2 - 5x + 6 = 0$.

(iv) A triangle has either 3-sides or 4-sides.

(v) 35 is a prime number or a composite number.

(vi) All prime integers are either even or odd.

(vii) $|x|$ is equal to either x or $-x$.

(viii) 6 is divisible by 2 and 3.

7. Rewrite each of the following statements in the form of conditional statements

(i) The square of an odd number is odd.

(ii) You will get a sweet dish after the dinner.

(iii) You will fail, if you will not study.

(iv) The unit digit of an integer is 0 or 5 if it is divisible by 5.

(v) The square of a prime number is not prime.

(vi) $2b = a + c$, if a, b and c are in A.P.

8. Form the biconditional statement $p \leftrightarrow q$, where

(i) p : The unit digit of an integer is zero.

q : It is divisible by 5.

(ii) p : A natural number n is odd.

q : Natural number n is not divisible by 2.

(iii) p : A triangle is an equilateral triangle.

q : All three sides of a triangle are equal.

9. Write down the contrapositive of the following statements:

(i) If $x = y$ and $y = 3$, then $x = 3$.

(ii) If n is a natural number, then n is an integer.

(iii) If all three sides of a triangle are equal, then the triangle is equilateral.

(iv) If x and y are negative integers, then xy is positive.

(v) If natural number n is divisible by 6, then n is divisible by 2 and 3.

(vi) If it snows, then the weather will be cold.

(vii) If x is a real number such that $0 < x < 1$, then $x^2 < 1$.

10. Write down the converse of following statements :

(i) If a rectangle 'R' is a square, then R is a rhombus.

(ii) If today is Monday, then tomorrow is Tuesday.

(iii) If you go to Agra, then you must visit Taj Mahal.

(iv) If the sum of squares of two sides of a triangle is equal to the square of third side of a triangle, then the triangle is right angled.

(v) If all three angles of a triangle are equal, then the triangle is equilateral.

(vi) If $x : y = 3 : 2$, then $2x = 3y$.

(vii) If S is a cyclic quadrilateral, then the opposite angles of S are supplementary.

(viii) If x is zero, then x is neither positive nor negative.

(ix) If two triangles are similar, then the ratio of their corresponding sides are equal.

11. Identify the Quantifiers in the following statements.

(i) There exists a triangle which is not equilateral.

(ii) For all real numbers x and y, $xy = yx$.

(iii) There exists a real number which is not a rational number.

(iv) For every natural number x, $x + 1$ is also a natural number.

(v) For all real numbers x with $x > 3$, x^2 is greater than 9.

(vi) There exists a triangle which is not an isosceles triangle.

(vii) For all negative integers x, x^3 is also a negative integers.

(viii) There exists a statement in above statements which is not true.

(ix) There exists a even prime number other than 2.

(x) There exists a real number x such that $x^2 + 1 = 0$.

12. Prove by direct method that for any integer 'n', $n^3 - n$ is always even.

[**Hint:** Two cases (i) n is even, (ii) n is odd.]

13. Check the validity of the following statement.

(i) p : 125 is divisible by 5 and 7.

(ii) q : 131 is a multiple of 3 or 11.

14. Prove the following statement by contradication method.

p : The sum of an irrational number and a rational number is irrational.

15. Prove by direct method that for any real numbers x, y if $x = y$, then $x^2 = y^2$.

16. Using contrapositive method prove that if n^2 is an even integer, then n is also an even integers.

Objective Type Questions

Choose the correct answer out of the four options given against each of the Exercises 17 to 36 (M.C.Q.).

17. Which of the following is a statement.

(A) x is a real number.

(B) Switch off the fan.

(C) 6 is a natural number.

(D) Let me go.

18. Which of the following is not a statement

(A) Smoking is injurious to health.

(B) $2 + 2 = 4$

(C) 2 is the only even prime number.

(D) Come here.

19. The connective in the statement

"$2 + 7 > 9$ or $2 + 7 < 9$" is

(A) and

(B) or

(C) $>$

(D) $<$

20. The connective in the statement

"Earth revolves round the Sun and Moon is a satellite of earth" is

(A) or

(B) Earth

(C) Sun

(D) and

21. The negation of the statement

"A circle is an ellipse" is

(A) An ellipse is a circle.

(B) An ellipse is not a circle.

(C) A circle is not an ellipse.

(D) A circle is an ellipse.

22. The negation of the statement

"7 is greater than 8" is

(A) 7 is equal to 8.

(B) 7 is not greater than 8.

(C) 8 is less than 7.

(D) none of these

23. The negation of the statement

"72 is divisible by 2 and 3" is

(A) 72 is not divisible by 2 or 72 is not divisible by 3.

(B) 72 is not divisible by 2 and 72 is not divisible by 3.

(C) 72 is divisible by 2 and 72 is not divisible by 3.

(D) 72 is not divisible by 2 and 72 is divisible by 3.

24. The negation of the statement

"Plants take in CO_2 and give out O_2" is

(A) Plants do not take in CO_2 and do not give out O_2.

(B) Plants do not take in CO_2 or do not give out O_2.

(C) Plants take in CO_2 and do not give out O_2.

(D) Plants take in CO_2 or do not give out O_2.

25. The negation of the statement

"Rajesh or Rajni lived in Bangalore" is

(A) Rajesh did not live in Bangalore or Rajni lives in Bangalore.

(B) Rajesh lives in Bangalore and Rajni did not live in Bangalore.

(C) Rajesh did not live in Bangalore and Rajni did not live in Bangalore.

(D) Rajesh did not live in Bangalore or Rajni did not live in Bangalore.

26. The negation of the statement

"101 is not a multiple of 3" is

(A) 101 is a multiple of 3.

(B) 101 is a multiple of 2.

(C) 101 is an odd number.

(D) 101 is an even number.

27. The contrapositive of the statement

"If 7 is greater than 5, then 8 is greater than 6" is

(A) If 8 is greater than 6, then 7 is greater than 5.

(B) If 8 is not greater than 6, then 7 is greater than 5.

(C) If 8 is not greater than 6, then 7 is not greater than 5.

(D) If 8 is greater than 6, then 7 is not greater than 5.

28. The converse of the statement

"If $x > y$, then $x + a > y + a$" is

(A) If $x < y$, then $x + a < y + a$.

(B) If $x + a > y + a$, then $x > y$.

(C) If $x < y$, then $x + a > y + a$.

(D) If $x > y$, then $x + a < y + a$.

29. The converse of the statement

"If sun is not shining, then sky is filled with clouds" is

(A) If sky is filled with clouds, then the sun is not shining.

(B) If sun is shining, then sky is filled with clouds.

(C) If sky is clear, then sun is shining.

(D) If sun is not shining, then sky is not filled with clouds.

30. The contrapositive of the statement

"If p, then q", is

(A) If q, then p.

(B) If p, then $\sim q$.

(C) If $\sim q$, then $\sim p$.

(D) If $\sim p$, then $\sim q$.

31. The statement

"If x^2 is not even, then x is not even" is converse of the statement

(A) If x^2 is odd, then x is even.

(B) If x is not even, then x^2 is not even.

(C) If x is even, then x^2 is even.

(D) If x is odd, then x^2 is even.

32. The contrapositive of statement

'If Chandigarh is capital of Punjab, then Chandigarh is in India' is

(A) If Chandigarh is not in India, then Chandigarh is not the capital of Punjab.

(B) If Chandigarh is in India, then Chandigarh is Capital of Punjab.

(C) If Chandigarh is not capital of Punjab, then Chandigarh is not capital of India.

(D) If Chandigarh is capital of Punjab, then Chandigarh is not in India.

33. Which of the following is the conditional $p \rightarrow q$?

(A) q is sufficient for p.

(B) p is necessary for q.

(C) p only if q.

(D) if q, then p.

34. The negation of the statement "The product of 3 and 4 is 9" is

(A) It is false that the product of 3 and 4 is 9.

(B) The product of 3 and 4 is 12.

(C) The product of 3 and 4 is not 12.

(D) It is false that the product of 3 and 4 is not 9.

35. Which of the following is not a negation of

"A natural number is greater than zero"

(A) A natural number is not greater than zero.

(B) It is false that a natural number is greater than zero.

(C) It is false that a natural number is not greater than zero.

(D) None of the above

36. Which of the following statement is a conjunction ?

(A) Ram and Shyam are friends.

(B) Both Ram and Shyam are tall.

(C) Both Ram and Shyam are enemies.

(D) None of the above.

37. State whether the following sentences are statements are not :

(i) The angles opposite to equal sides of a triangle are equal.

(ii) The moon is a satellite of earth.

(iii) May God bless you!

(iv) Asia is a continent.

(v) How are you?

Chapter 15

STATISTICS

15.1 Overview

In earlier classes, you have studied measures of central tendency such as mean, mode, median of ungrouped and grouped data. In addition to these measures, we often need to calculate a second type of measure called a **measure of dispersion** which measures the **variation** in the observations about the **middle value**– mean or median etc.

This chapter is concerned with some important measures of dispersion such as mean deviation, variance, standard deviation etc., and finally analysis of frequency distributions.

15.1.1 *Measures of dispersion*

(a) **Range**The measure of dispersion which is easiest to understand and easiest to calculate is the **range**. Range is defined as:

Range = Largest observation – Smallest observation

(b) **Mean Deviation**

(i) Mean deviation for ungrouped data:

For n observation x_1, x_2, ..., x_n, the **mean deviation about their mean** $\overline{x}$ is given by

$$\text{M.D}\,(\overline{x}) = \frac{\sum |x_i - \overline{x}|}{n} \qquad (1)$$

Mean deviation about their median M is given by

$$\text{M.D (M)} = \frac{\sum |x_i - \text{M}|}{n} \qquad (2)$$

(ii) Mean deviation for discrete frequency distribution

Let the given data consist of discrete observations $x_1, x_2, \ldots, x_n$ occurring with frequencies $f_1, f_2, \ldots, f_n$, respectively. In this case

$$\text{M.D}(\bar{x}) = \frac{\sum f_i |x_i - \bar{x}|}{\sum f_i} = \frac{\sum f_i |x_i - \bar{x}|}{\text{N}} \quad (3)$$

$$\text{M.D (M)} = \frac{\sum f_i |x_i - \text{M}|}{\text{N}} \quad (4)$$

where $\text{N} = \sum f_i$.

(iii) Mean deviation for continuous frequency distribution (Grouped data).

$$\text{M.D}(\bar{x}) = \frac{\sum f_i |x_i - \bar{x}|}{\text{N}} \quad (5)$$

$$\text{M.D (M)} = \frac{\sum f_i |x_i - \text{M}|}{\text{N}} \quad (6)$$

where x_i are the midpoints of the classes, $\bar{x}$ and M are, respectively, the mean and median of the distribution.

(c) **Variance :** Let $x_1, x_2, ..., x_n$ be n observations with $\bar{x}$ as the mean. The variance, denoted by σ^2, is given by

$$\sigma^2 = \frac{1}{n}\sum (x_i - \bar{x})^2 \quad (7)$$

(d) **Standard Deviation:** If σ^2 is the variance, then σ, is called the standard deviation, is given by

$$\sigma = \sqrt{\frac{1}{n}\sum (x_i - \bar{x})^2} \quad (8)$$

(e) **Standard deviation for a discrete frequency distribution** is given by

$$\sigma = \sqrt{\frac{1}{\text{N}}\sum f_i (x_i - \bar{x})^2} \quad (9)$$

where f_i's are the frequencies of x_i' s and $\text{N} = \sum_{i=1}^{n} f_i$.

(f) **Standard deviation of a continuous frequency distribution (grouped data)** is given by

$$\sigma = \sqrt{\frac{1}{N}\sum f_i (x_i - \overline{x})^2} \qquad (10)$$

where x_i are the midpoints of the classes and f_i their respective frequencies. Formula (10) is same as

$$\sigma = \frac{1}{N}\sqrt{N\sum f_i x_i^2 - \left(\sum f_i x_i\right)^2} \qquad (11)$$

(g) Another formula for standard deviation :

$$\sigma_x = \frac{h}{N}\sqrt{N\sum f_i y_i^2 - \left(\sum f_i y_i\right)^2} \qquad (12)$$

where h is the width of class intervals and $y_i = \frac{x_i - A}{h}$ and A is the assumed mean.

15.1.2 ***Coefficient of variation*** It is sometimes useful to describe **variability** by expressing the standard deviation as a proportion of mean, usually a percentage. The formula for it as a percentage is

$$\text{Coefficient of variation} = \frac{\text{Standard deviation}}{\text{Mean}} \times 100$$

15.2 Solved Examples

Short Answer Type

Example 1 Find the mean deviation about the mean of the following data:

Size (x):	1	3	5	7	9	11	13	15
Frequency (f):	3	3	4	14	7	4	3	4

Solution Mean $= \overline{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{3+9+20+98+63+44+39+60}{42} = \frac{336}{42} = 8$

$$\text{M.D. } (\overline{x}) = \frac{\sum f_i |x_i - \overline{x}|}{\sum f_i} = \frac{3(7)+3(5)+4(3)+14(1)+7(1)+4(3)+3(5)+4(7)}{42}$$

$$= \frac{21+15+12+14+7+12+15+28}{42} = \frac{62}{21} = 2.95$$

Example 2 Find the variance and standard deviation for the following data:
57, 64, 43, 67, 49, 59, 44, 47, 61, 59

Solution Mean ($\bar{x}$) $= \dfrac{57+64+43+67+49+59+61+59+44+47}{10} = \dfrac{550}{10} = 55$

$$\text{Variance } (\sigma^2) = \frac{\sum (x_i - \bar{x})^2}{n}$$

$$= \frac{2^2+9^2+12^2+12^2+6^2+4^2+6^2+4^2+11^2+8^2}{10}$$

$$= \frac{662}{10} = 66.2$$

Standard deviation (σ) $= \sqrt{\sigma^2} = \sqrt{66.2} = 8.13$

Example 3 Show that the two formulae for the standard deviation of ungrouped data.

$$\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}} \qquad \text{and} \qquad \sigma' = \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2}$$

are equivalent.

Solution We have $\sum (x_i - \bar{x})^2 = \sum (x_i^2 - 2\bar{x}\, x_i + \bar{x}^2)$

$$= \sum x_i^2 + \sum -2\,\bar{x}\, x_i + \sum \bar{x}^2$$

$$= \sum x_i^2 - 2\bar{x} \sum x_i + (\bar{x})^2 \sum 1$$

$$= \sum x_i^2 - 2\bar{x}\,(n\bar{x}) + n\,\bar{x}^2$$

$$= \sum x_i^2 - n\bar{x}^2$$

Dividing both sides by n and taking their square root, we get $\sigma = \sigma'$.

Example 4 Calculate **variance** of the following data :

Class interval	Frequency
4 - 8	3
8 - 12	6
12 - 16	4
16 - 20	7

$$\text{Mean } (\bar{x}) = \frac{\sum f_i x_i}{\sum f_i} = \frac{3 \times 6 + 6 \times 10 + 4 \times 14 + 7 \times 18}{20} = 13$$

Solution Variance (σ^2) = $\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i} = \frac{3(-7)^2 + 6(-3)^2 + 4(1)^2 + 7(5)^2}{20}$

$$= \frac{147 + 54 + 4 + 175}{20} = 19$$

Long Answer Type

Example 5 Calculate mean, variation and standard deviation of the following frequency distribution:

Classes	Frequency
1 - 10	11
10 - 20	29
20 - 30	18
30 - 40	4
40 - 50	5
50 - 60	3

Solution Let A, the assumed mean, be 25.5. Here $h = 10$

Classes	x_i	$y_i = \dfrac{x_i - 25.5}{10}$	f_i	$f_i\, y_i$	$f_i\, y_i^2$
1 - 10	5.5	–2	11	–22	44
10 - 20	15.5	–1	29	–29	29
20 - 30	25.5	0	18	0	0
30 - 40	35.5	1	4	4	4
40 - 50	45.5	2	5	10	20
50 - 60	55.5	3	3	9	27
			70	**–28**	**124**

$$x' = \frac{\sum f_i y_i}{\sum f_i} = \frac{-28}{70} = -0.4$$

Mean $= \bar{x} = 25.5 + (-10)(0.4) = 21.5$

Variance $$(\sigma^2) = \left[\frac{h}{N}\sqrt{N\sum f_i y_i^2 - \left(\sum f_i y_i\right)^2}\right]^2$$

$$= \frac{10 \times 10}{70 \times 70}\,[70(124) - (-28)^2]$$

$$= \frac{70\,(124)}{7 \times 7} - \frac{28 \times 28}{7 \times 7} = \frac{1240}{7} - 16 = 161$$

$$\text{S.D. } (\sigma) = \sqrt{161} = 12.7$$

Example 6 Life of bulbs produced by two factories A and B are given below:

Length of life (in hours)	Factory A (Number of bulbs)	Factory B (Number of bulbs)
550 - 650	10	8
650 - 750	22	60
750 - 850	52	24
850 - 950	20	16
950 - 1050	16	12
	120	**120**

The bulbs of which factory are more consistent from the point of view of length of life?

Solution Here $h = 100$, let A (assumed mean) = 800.

Length of life (in hour)	Mid values(x_i)	$y_i = \frac{x_i - A}{10}$	Factory A			Factory B		
			f_i	$f_i y_i$	$f_i y_i^2$	f_i	$f_i y_i$	$f_i y_i^2$
550 - 650	600	–2	10	–20	40	8	–16	32
650 - 750	700	–1	22	–22	22	60	– 60	60
750 - 850	800	0	52	0	0	24	0	0
850 - 950	900	1	20	20	20	16	16	16
950 - 1050	1000	2	16	32	64	12	24	48
			120	**10**	**146**	**120**	**–36**	**156**

For factory A

$$\text{Mean } (\bar{x}) = 800 + \frac{10}{120} \times 100 = 816.67 \text{ hours}$$

$$\text{S.D.} = \frac{100}{120}\sqrt{120\,(146) - 100} = 109.98$$

Therefore, Coefficient of variation (C.V.) = $\frac{\text{S.D.}}{\bar{x}} \times 100 = \frac{109.98}{816.67} \times 100 = 13.47$

For factory B

$$\text{Mean} = 800 + \left(\frac{-36}{120}\right) 100 = 770$$

$$\text{S.D.} = \frac{100}{120}\sqrt{120\,(156) - (-36)^2} = 110$$

Therefore, Coefficient of variation $= \frac{\text{S.D.}}{\text{Mean}} \times 100 = \frac{110}{770} \times 100 = 14.29$

Since C.V. of factory B > C.V. of factory A $\Rightarrow$ Factory B has more variability which means bulbs of factory A are more consistent.

Objective Type Questions

Choose the correct answer out of the four options given against each of the Examples 7 to 9 (M.C.Q.).

Example 7 The mean deviation of the data 2, 9, 9, 3, 6, 9, 4 from the mean is

(A) 2.23 (B) 2.57 (C) 3.23 (D) 3.57

Solution (B) is the correct answer

$$\text{M.D.}\,(\bar{x}) = \frac{\sum |x_i - \bar{x}|}{n} = \frac{4+3+3+3+0+3+2}{7} = 2.57$$

Example 8 Variance of the data 2, 4, 5, 6, 8, 17 is 23.33. Then variance of 4, 8, 10, 12, 16, 34 will be

(A) 23.23 (B) 25.33 (C) 46.66 (D) 48.66

Solution (C) is the correct answer. When each observation is multiplied by 2, then variance is also multiplied by 2.

Example 9 A set of n values $x_1, x_2, ..., x_n$ has standard deviation 6. The standard deviation of n values $x_1 + k, x_2 + k, ..., x_n + k$ will be

(A) σ (B) $\sigma + k$ (C) $\sigma - k$ (D) $k\sigma$

Solution (A) is correct answer. If each observation is increased by a constant k, then standard deviation is unchanged.

15.3 EXERCISE

Short Answer Type

1. Find the mean deviation about the mean of the distribution:

Size	20	21	22	23	24
Frequency	6	4	5	1	4

2. Find the mean deviation about the median of the following distribution:

Marks obtained	10	11	12	14	15
No. of students	2	3	8	3	4

3. Calculate the mean deviation about the mean of the set of first n natural numbers when n is an odd number.

4. Calculate the mean deviation about the mean of the set of first n natural numbers when n is an even number.

5. Find the standard deviation of the first n natural numbers.

6. The mean and standard deviation of some data for the time taken to complete a test are calculated with the following results:

Number of observations = 25, mean = 18.2 seconds, standard deviation = 3.25 seconds.

Further, another set of 15 observations $x_1, x_2, ..., x_{15}$, also in seconds, is now available and we have $\sum_{i=1}^{15} x_i = 279$ and $\sum_{i=1}^{15} x_i^2 = 5524$. Calculate the standard derivation based on all 40 observations.

7. The mean and standard deviation of a set of n_1 observations are $\bar{x}_1$ and s_1, respectively while the mean and standard deviation of another set of n_2 observations are $\bar{x}_2$ and s_2, respectively. Show that the standard deviation of the combined set of $(n_1 + n_2)$ observations is given by

$$\text{S.D.} = \sqrt{\frac{n_1(s_1)^2 + n_2(s_2)^2}{n_1 + n_2} + \frac{n_1 n_2(\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2}}$$

8. Two sets each of 20 observations, have the same standard derivation 5. The first set has a mean 17 and the second a mean 22. Determine the standard deviation of the set obtained by combining the given two sets.

9. The frequency distribution:

x	A	2A	3A	4A	5A	6A
f	2	1	1	1	1	1

where A is a positive integer, has a variance of 160. Determine the value of A.

10. For the frequency distribution:

x	2	3	4	5	6	7
f	4	9	16	14	11	6

Find the standard distribution.

11. There are 60 students in a class. The following is the frequency distribution of the marks obtained by the students in a test:

Marks	0	1	2	3	4	5
Frequency	$x - 2$	x	x^2	$(x + 1)^2$	$2x$	$x + 1$

where x is a positive integer. Determine the mean and standard deviation of the marks.

12. The mean life of a sample of 60 bulbs was 650 hours and the standard deviation was 8 hours. A second sample of 80 bulbs has a mean life of 660 hours and standard deviation 7 hours. Find the overall standard deviation.

13. Mean and standard deviation of 100 items are 50 and 4, respectively. Find the sum of all the item and the sum of the squares of the items.

14. If for a distribution $\sum(x-5)=3$, $\sum(x-5)^2=43$ and the total number of item is 18, find the mean and standard deviation.

15. Find the mean and variance of the frequency distribution given below:

x	$1 \le x < 3$	$3 \le x < 5$	$5 \le x < 7$	$7 \le x < 10$
f	6	4	5	1

Long Answer Type

16. Calculate the mean deviation about the mean for the following frequency distribution:

Class interval	0 - 4	4 - 8	8 - 12	12 - 16	16 - 20
Frequency	4	6	8	5	2

17. Calculate the mean deviation from the median of the following data:

Class interval	0 - 6	6 - 12	12 - 18	18 - 24	24 - 30
Frequency	4	5	3	6	2

18. Determine the mean and standard deviation for the following distribution:

Marks	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Frequency	1	6	6	8	8	2	2	3	0	2	1	0	0	0	1

19. The weights of coffee in 70 jars is shown in the following table:

Weight (in grams)	**Frequency**
200 - 201	13
201 - 202	27
202 - 203	18
203 - 204	10
204 - 205	1
205 - 206	1

Determine variance and standard deviation of the above distribution.

20. Determine mean and standard deviation of first n terms of an A.P. whose first term is a and common difference is d.

21. Following are the marks obtained, out of 100, by two students Ravi and Hashina in 10 tests.

Ravi	25	50	45	30	70	42	36	48	35	60
Hashina	10	70	50	20	95	55	42	60	48	80

Who is more intelligent and who is more consistent?

22. Mean and standard deviation of 100 observations were found to be 40 and 10, respectively. If at the time of calculation two observations were wrongly taken as 30 and 70 in place of 3 and 27 respectively, find the correct standard deviation.

23. While calculating the mean and variance of 10 readings, a student wrongly used the reading 52 for the correct reading 25. He obtained the mean and variance as 45 and 16 respectively. Find the correct mean and the variance.

Objective Type Questions

Choose the correct answer out of the given four options in each of the Exercises 24 to 39 (M.C.Q.).

24. The mean deviation of the data 3, 10, 10, 4, 7, 10, 5 from the mean is

(A) 2 (B) 2.57 (C) 3 (D) 3.75

25. Mean deviation for n observations $x_1, x_2, ..., x_n$ from their mean $\bar{x}$ is given by

(A) $\sum_{i=1}^{n}(x_i - \bar{x})$ (B) $\frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$

(C) $\sum_{i=1}^{n}(x_i - \bar{x})^2$ (D) $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

26. When tested, the lives (in hours) of 5 bulbs were noted as follows:
1357, 1090, 1666, 1494, 1623

The mean deviations (in hours) from their mean is

(A) 178 (B) 179 (C) 220 (D) 356

27. Following are the marks obtained by 9 students in a mathematics test:
50, 69, 20, 33, 53, 39, 40, 65, 59

The mean deviation from the median is:

(A) 9 (B) 10.5 (C) 12.67 (D) 14.76

28. The standard deviation of the data 6, 5, 9, 13, 12, 8, 10 is

(A) $\sqrt{\frac{52}{7}}$ (B) $\frac{52}{7}$ (C) $\sqrt{6}$ (D) 6

29. Let $x_1, x_2, ..., x_n$ be n observations and $\overline{x}$ be their arithmetic mean. The formula for the standard deviation is given by

(A) $\sum (x_i - \overline{x})^2$ (B) $\frac{\sum (x_i - \overline{x})^2}{n}$

(C) $\sqrt{\frac{\sum (x_i - \overline{x})^2}{n}}$ (D) $\sqrt{\frac{\sum x_i^2}{n} + \overline{x}^2}$

30. The mean of 100 observations is 50 and their standard deviation is 5. The sum of all squares of all the observations is

(A) 50000 (B) 250000 (C) 252500 (D) 255000

31. Let a, b, c, d, e be the observations with mean m and standard deviation s. The standard deviation of the observations $a + k, b + k, c + k, d + k, e + k$ is

(A) s (B) ks (C) $s + k$ (D) $\frac{s}{k}$

32. Let x_1, x_2, x_3, x_4, x_5 be the observations with mean m and standard deviation s. The standard deviation of the observations $kx_1, kx_2, kx_3, kx_4, kx_5$ is

(A) $k + s$ (B) $\frac{s}{k}$ (C) ks (D) s

33. Let $x_1, x_2, ... x_n$ be n observations. Let $w_i = lx_i + k$ for $i = 1, 2, ...n$, where l and k are constants. If the mean of x_i's is 48 and their standard deviation is 12, the mean of w_i's is 55 and standard deviation of w_i's is 15, the values of l and k should be

(A) $l = 1.25, k = -5$ (B) $l = -1.25, k = 5$

(C) $l = 2.5, k = -5$ (D) $l = 2.5, k = 5$

34. Standard deviations for first 10 natural numbers is

(A) 5.5 (B) 3.87 (C) 2.97 (D) 2.87

35. Consider the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. If 1 is added to each number, the variance of the numbers so obtained is

(A) 6.5 (B) 2.87 (C) 3.87 (D) 8.25

36. Consider the first 10 positive integers. If we multiply each number by –1 and then add 1 to each number, the variance of the numbers so obtained is

(A) 8.25 (B) 6.5 (C) 3.87 (D) 2.87

37. The following information relates to a sample of size 60: $\sum x^2 = 18000$,

$\sum x = 960$

The variance is

(A) 6.63 (B) 16 (C) 22 (D) 44

38. Coefficient of variation of two distributions are 50 and 60, and their arithmetic means are 30 and 25 respectively. Difference of their standard deviation is

(A) 0 (B) 1 (C) 1.5 (D) 2.5

39. The standard deviation of some temperature data in °C is 5. If the data were converted into ºF, the variance would be

(A) 81 (B) 57 (C) 36 (D) 25

Fill in the blanks in Exercises from 40 to 46.

40. Coefficient of variation = $\dfrac{\ldots}{\text{Mean}} \times 100$

41. If $\overline{x}$ is the mean of n values of x, then $\sum_{i=1}^{n}(x_i - \overline{x})$ is always equal to _______.

If a has any value other than $\overline{x}$, then $\sum_{i=1}^{n}(x_i - \overline{x})^2$ is __________ than

$\sum (x_i - a)^2$

42. If the variance of a data is 121, then the standard deviation of the data is _______.

43. The standard deviation of a data is ____________ of any change in orgin, but is _____ on the change of scale.

44. The sum of the squares of the deviations of the values of the variable is _______ when taken about their arithmetic mean.

45. The mean deviation of the data is _______ when measured from the median.

46. The standard deviation is _______ to the mean deviation taken from the arithmetic mean.

Chapter 16

PROBABILITY

16.1 Overview

Probability is defined as a quantitative measure of uncertainty – a numerical value that conveys the strength of our belief in the occurrence of an event. The probability of an event is always a number between 0 and 1 both 0 and 1 inclusive. If an event's probability is nearer to 1, the higher is the likelihood that the event will occur; the closer the event's probability to 0, the smaller is the likelihood that the event will occur. If the event cannot occur, its probability is 0. If it must occur (i.e., its occurrence is certain), its probability is 1.

16.1.1 ***Random experiment*** An experiment is random means that the experiment has more than one possible outcome and it is not possible to predict with certainty which outcome that will be. For instance, in an experiment of tossing an ordinary coin, it can be predicted with certainty that the coin will land either heads up or tails up, but it is not known for sure whether heads or tails will occur. If a die is thrown once, any of the six numbers, i.e., 1, 2, 3, 4, 5, 6 may turn up, not sure which number will come up.

(i) **Outcome** A possible result of a random experiment is called its **outcome** for example if the experiment consists of tossing a coin twice, some of the outcomes are HH, HT etc.

(ii) **Sample Space** A sample space is the set of all possible outcomes of an experiment. In fact, it is the universal set S pertinent to a given experiment.

The sample space for the experiment of tossing a coin twice is given by

$$S = \{HH, HT, TH, TT\}$$

The sample space for the experiment of drawing a card out of a deck is the set of all cards in the deck.

16.1.2 ***Event*** An event is a subset of a sample space S. For example, the event of drawing an ace from a deck is

A = {Ace of Heart, Ace of Club, Ace of Diamond, Ace of Spade}

16.1.3 ***Types of events***

(i) **Impossible and Sure Events** The empty set ϕ and the sample space S describe events. In fact ϕ is called an impossible event and S, i.e., the whole sample space is called a sure event.

(ii) **Simple or Elementary Event** If an event E has only one sample point of a sample space, i.e., a single outcome of an experiment, it is called a **simple** or **elementary** event. The sample space of the experiment of tossing two coins is given by

$$S = \{HH, HT, TH, TT\}$$

The event $E_1 = \{HH\}$ containing a single outcome HH of the sample space S is a simple or elementary event. If one card is drawn from a well shuffled deck, any particular card drawn like 'queen of Hearts' is an elementary event.

(iii) **Compound Event** If an event has more than one sample point it is called a **compound event**, for example, S = {HH, HT} is a compound event.

(iv) **Complementary event** Given an event A, the complement of A is the event consisting of all sample space outcomes that do not correspond to the occurrence of A.

The complement of A is denoted by A' or $\overline{A}$. It is also called the event 'not A'. Further $P(\overline{A})$ denotes the probability that A will not occur.

$$A' = \overline{A} = S - A = \{w : w \in S \text{ and } w \notin A\}$$

16.1.4 ***Event 'A or B'*** If A and B are two events associated with same sample space, then the event 'A or B' is same as the event $A \cup B$ and contains all those elements which are either in A or in B or in both. Further more, $P(A \cup B)$ denotes the probability that A or B (or both) will occur.

16.1.5 ***Event 'A and B'*** If A and B are two events associated with a sample space, then the event 'A and B' is same as the event $A \cap B$ and contains all those elements which are common to both A and B. Further more, $P(A \cap B)$ denotes the probability that both A and B will simultaneously occur.

16.1.6 ***The Event 'A but not B' (Difference A – B)*** An event A – B is the set of all those elements of the same space S which are in A but not in B, i.e., $A - B = A \cap B'$.

16.1.7 ***Mutually exclusive*** Two events A and B of a sample space S are **mutually exclusive** if the occurrence of any one of them excludes the occurrence of the other event. Hence, the two events A and B cannot occur simultaneously, and thus $P(A \cap B) = 0$.

Remark Simple or elementary events of a sample space are always mutually exclusive. For example, the elementary events {1}, {2}, {3}, {4}, {5} or {6} of the experiment of throwing a dice are mutually exclusive.

Consider the experiment of throwing a die once.

The events E = getting a even number and F = getting an odd number are mutually exclusive events because $E \cap F = \phi$.

Note For a given sample space, there may be two or more mutually exclusive events.

16.1.8 ***Exhaustive events*** If $E_1, E_2, ..., E_n$ are n events of a sample space S and if

$$E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = \bigcup_{i=1}^{n} E_i = S$$

then $E_1, E_2, ..., E_n$ are called exhaustive events.

In other words, events $E_1, E_2, ..., E_n$ of a sample space S are said to be exhaustive if atleast one of them necessarily occur whenever the experiment is performed.

Consider the example of rolling a die. We have S = {1, 2, 3, 4, 5, 6}. Define the two events A : 'a number less than or equal to 4 appears.'

B : 'a number greater than or equal to 4 appears.'

Now A : {1, 2, 3, 4}, B = {4, 5, 6}

$A \cup B = \{1, 2, 3, 4, 5, 6\} = S$

Such events A and B are called exhaustive events.

16.1.9 ***Mutually exclusive and exhaustive events*** If $E_1, E_2, ..., E_n$ are n events of a sample space S and if $E_i \cap E_j = \phi$ for every $i \neq j$, i.e., E_i and E_j are pairwise disjoint and $\bigcup_{i=1}^{n} E_i = S$, then the events $E_1, E_2, ..., E_n$ are called mutually exclusive and exhaustive events.

Consider the example of rolling a die.

We have S = {1, 2, 3, 4, 5, 6}

Let us define the three events as

A = a number which is a perfect square

B = a prime number

C = a number which is greater than or equal to 6

Now A = {1, 4}, B = {2, 3, 5}, C = {6}

Note that $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\} = S$. Therefore, A, B and C are exhaustive events.

Also $A \cap B = B \cap C = C \cap A = \phi$

Hence, the events are pairwise disjoint and thus mutually exclusive.

Classical approach is useful, when all the outcomes of the experiment are equally likely. We can use logic to assign probabilities. To understand the classical method consider the experiment of tossing a fair coin. Here, there are two equally likely outcomes - head (H) and tail (T). When the elementary outcomes are taken as equally likely, we have a uniform probablity model. If there are k elementary outcomes in S, each is assigned the probability of $\frac{1}{k}$. Therefore, logic suggests that the probability of observing a head, denoted by P (H), is $\frac{1}{2} = 0.5$, and that the probability of observing a tail,denoted P (T), is also $\frac{1}{2} = 5$. Notice that each probability is between 0 and 1. Further H and T are all the outcomes of the experiment and P (H) + P (T) = 1.

16.1.10 ***Classical definition*** If all of the outcomes of a sample space are equally likely, then the probability that an event will occur is equal to the ratio :

$$\frac{\text{The number of outcomes favourable to the event}}{\text{The total number of outcomes of the sample space}}$$

Suppose that an event E can happen in h ways out of a total of n possible equally likely ways.

Then the classical probability of occurrence of the event is denoted by

$$P\,(E) = \frac{h}{n}$$

The probability of non occurrence of the event E is denoted by

$$P\,(\text{not } E) = \frac{n-h}{n} = 1 - \frac{h}{n} = 1 - P(E)$$

Thus $\qquad P\,(E) + P\,(\text{not } E) = 1$

The event 'not E' is denoted by $\overline{E}$ or E′ (complement of E)

Therefore $\qquad P\,(\overline{E}) = 1 - P\,(E)$

16.1.11 ***Axiomatic approach to probability*** Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S, i.e., P (S) and range is the interval [0, 1] i.e. P : P (S) $\rightarrow$ [0, 1] satisfying the following axioms.

(i) For any event E, P (E) $\geq$ 0.

(ii) P (S) = 1

(iii) If E and F are mutually exclusive events, then P (E $\cup$ F) = P (E) + P (F).

It follows from (iii) that P (ϕ) = 0.

Let S be a sample space containing elementary outcomes $w_1, w_2, ..., w_n$,

i.e., S = $\{w_1, w_2, ..., w_n\}$

It follows from the axiomatic definition of probability that

(i) $0 \le P(w_i) \le 1$ for each $w_i \in S$

(ii) $P(w_1) + P(w_2) + ... + P(w_n) = 1$

(iii) P (A) = $\sum P(w_i)$ for any event A containing elementary events w_i.

For example, if a fair coin is tossed once

$$P(H) = P(T) = \frac{1}{2}$$ satisfies the three axioms of probability.

Now suppose the coin is not fair and has double the chances of falling heads up as compared to the tails, then P (H) = $\frac{2}{3}$ and P (T) = $\frac{1}{3}$.

This assignment of probabilities are also valid for H and T as these satisfy the axiomatic definitions.

16.1.12 ***Probabilities of equally likely outcomes*** Let a sample space of an experiment be S = $\{w_1, w_2, ..., w_n\}$ and suppose that all the outcomes are equally likely to occur i.e., the chance of occurrence of each simple event must be the same

i.e., $P(w_i) = p$ for all $w_i \in S$, where $0 \le p \le 1$

Since $$\sum_{i=1}^{n} P(w_i) = 1$$

i.e., $p + p + p + ... + p$ (n times) = 1

$$\Rightarrow \quad np = 1, \quad \text{i.e.} \quad p = \frac{1}{n}$$

Let S be the sample space and E be an event, such that $n(S) = n$ and $n(E) = m$. If each outcome is equally likely, then it follows that

$$P(E) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to E}}{\text{Total number of possible outcomes}}$$

16.1.13 ***Addition rule of probability*** If A and B are any two events in a sample space S, then the probability that atleast one of the events A or B will occur is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Similarly, for three events A, B and C, we have

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

16.1.14 ***Addition rule for mutually exclusive events*** If A and B are disjoint sets, then

$P(A \cup B) = P(A) + P(B)$ [since $P(A \cap B) = P(\phi) = 0$, where A and B are disjoint].

The addition rule for mutually exclusive events can be extended to more than two events.

16.2 Solved Examples

Short Answer Type (S.A.)

Example 1 An ordinary deck of cards contains 52 cards divided into four suits. The red suits are diamonds and hearts and black suits are clubs and spades. The cards J, Q, and K are called face cards. Suppose we pick one card from the deck at random.

(a) What is the sample space of the experiment?

(b) What is the event that the chosen card is a black face card?

Solution

(a) The outcomes in the sample space S are 52 cards in the deck.

(b) Let E be the event that a black face card is chosen. The outcomes in E are Jack, Queen, King or spades or clubs. Symbolically

E = {J, Q, K, of spades and clubs} or E = {J♣, Q♣, K♣, J♠, Q♠, K♠}

Example 2 Suppose that each child born is equally likely to be a boy or a girl. Consider a family with exactly three children.

(a) List the eight elements in the sample space whose outcomes are all possible genders of the three children.

(b) Write each of the following events as a set and find its probability :

(i) The event that exactly one child is a girl.

(ii) The event that at least two children are girls

(iii) The event that no child is a girl

Solution

(a) All possible genders are expressed as :

S = {BBB, BBG, BGB, BGG, GBB,GBG, GGB, GGG}

(b) (i)Let A denote the event : 'exactly one child is a girl'

A = {BBG, BGB, GBB}

$$P(A) = \frac{3}{8}$$

(ii) Let B denote the event that at least two children are girls.

$$B = \{GGB, GBG, BGG, GGG\},\ P(B) = \frac{4}{8}.$$

(iii) Let C denote the event : 'no child is a girl'.

C = {BBB}

$$\therefore\ P(C) = \frac{1}{8}$$

Example 3

(a) How many two-digit positive integers are multiples of 3?

(b) What is the probability that a randomly chosen two-digit positive integer is a multiple of 3?

Solution

(a) 2 digit positive integers which are multiples of 3 are 12, 15, 18, ... , 99. Thus, there are 30 such integers.

(b) 2-digit positive integers are 10, 11, 12, ... , 99. Thus, there are 90 such numbers. Since out of these, 30 numbers are multiple of 3, therefore, the probability that a randomly chosen positive 2-digit integer is a multiple of 3, is $\frac{30}{90} = \frac{1}{3}$.

Example 4 A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits. If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?

Solution A PIN is a sequence of four symbols selected from 36 (26 letters + 10 digits) symbols.

By the fundamental principle of counting, there are $36 \times 36 \times 36 \times 36 = 36^4 = 1,679,616$ PINs in all. When repetition is not allowed the multiplication rule can be applied to conclude that there are

$$36 \times 35 \times 34 \times 33 = 1,413,720 \text{ different PINs}$$

The number of PINs that contain at least one repeated symbol

= 1,679,616 – 1,413,720 = 2,65,896

Thus, the probability that a randomly chosen PIN contains a repeated symbol is

$$\frac{265,896}{1,679,616} = .1583$$

Example 5 An experiment has four possible outcomes A, B, C and D, that are mutually exclusive. Explain why the following assignments of probabilities are not permissible:

(a) P(A) = .12, P (B) = .63, P (C) = 0.45, P (D) = – 0.20

(b) $P(A) = \frac{9}{120}$, $P(B) = \frac{45}{120}$ $P(C) = \frac{27}{120}$ $P(D) = \frac{46}{120}$

Solution

(a) Since P(D) = – 0.20, this is not possible as $0 \le P(A) \le 1$ for any event A.

(b) $P(S) = P(A \cup B \cup C \cup D) = \frac{9}{120} + \frac{45}{120} + \frac{27}{120} + \frac{46}{120} = \frac{127}{120} \neq 1.$

This violates the condition that P (S) = 1.

Example 6 Probability that a truck stopped at a roadblock will have faulty brakes or badly worn tires are 0.23 and 0.24, respectively. Also, the probability is 0.38 that a truck stopped at the roadblock will have faulty brakes and/or badly working tires. What is the probability that a truck stopped at this roadblock will have faulty breaks as well as badly worn tires?

Solution Let B be the event that a truck stopped at the roadblock will have faulty brakes and T be the event that it will have badly worn tires. We have P (B) = 0.23, P (T) = 0.24 and $P(B \cup T) = 0.38$

and $P(B \cup T) = P(B) + P(T) - P(B \cap T)$

So $0.38 = 0.23 + 0.24 - P(B \cap T)$

$\Rightarrow$ $P(B \cap T) = 0.23 + 0.24 - 0.38 = 0.09$

Example 7 If a person visits his dentist, suppose the probability that he will have his teeth cleaned is 0.48, the probability that he will have a cavity filled is 0.25, the probability that he will have a tooth extracted is 0.20, the probability that he will have a teeth cleaned and a cavity filled is 0.09, the probability that he will have his teeth cleaned and a tooth extracted is 0.12, the probability that he will have a cavity filled and a tooth extracted is 0.07, and the probability that he will have his teeth cleaned, a cavity filled, and a tooth extracted is 0.03. What is the probability that a person visiting his dentist will have atleast one of these things done to him?

Solution Let C be the event that the person will have his teeth cleaned and F and E be the event of getting cavity filled or tooth extracted, respectively. We are given

$$P(C) = 0.48, \quad P(F) = 0.25, \quad P(E) = .20, \quad P(C \cap F) = .09,$$

$$P(C \cap E) = 0.12, \quad P(E \cap F) = 0.07 \quad \text{and} \quad P(C \cap F \cap E) = 0.03$$

Now,
$$\begin{aligned} P(C \cup F \cup E) &= P(C) + P(F) + P(E) \\ &\quad - P(C \cap F) - P(C \cap E) - P(F \cap E) \\ &\quad + P(C \cap F \cap E) \\ &= 0.48 + 0.25 + 0.20 - 0.09 - 0.12 - 0.07 + 0.03 \\ &= 0.68 \end{aligned}$$

Long Answer Type

Example 8 An urn contains twenty white slips of paper numbered from 1 through 20, ten red slips of paper numbered from 1 through 10, forty yellow slips of paper numbered from 1 through 40, and ten blue slips of paper numbered from 1 through 10. If these 80 slips of paper are thoroughly shuffled so that each slip has the same probability of being drawn. Find the probabilities of drawing a slip of paper that is

(a) blue or white

(b) numbered 1, 2, 3, 4 or 5

(c) red or yellow and numbered 1, 2, 3 or 4

(d) numbered 5, 15, 25, or 35;

(e) white and numbered higher than 12 or yellow and numbered higher than 26.

Solution

(a) P (Blue or White) = P (Blue) + P (White) (Why?)

$$= \frac{10}{80} + \frac{20}{80} = \frac{30}{80} = \frac{3}{8}$$

(b) P (numbered 1, 2, 3, 4 or 5)

= P (1 of any colour) + P (2 of any colour)

+ P (3 of any colour) + P (4 of any colour) + P (5 of any colour)

$$= \frac{4}{80} + \frac{4}{80} + \frac{4}{80} + \frac{4}{80} + \frac{4}{80} = \frac{20}{80} = \frac{2}{8} = \frac{1}{4}$$

(c) P (Red or yellow and numbered 1, 2, 3 or 4)

= P (Red numbered 1, 2, 3 or 4) + P (yellow numbered 1, 2, 3 or 4)

$$= \frac{4}{80} + \frac{4}{80} = \frac{8}{80} = \frac{1}{10}$$

(d) P (numbered 5, 15, 25 or 35)

= P (5) + P (15) + P (25) + P (35)

= P (5 of White, Red, Yellow, Blue) + P (15 of White, Yellow) + P (25 of Yellow) + P (35 of Yellow)

$$= \frac{4}{80} + \frac{2}{80} + \frac{1}{80} + \frac{1}{80} = \frac{8}{80} = \frac{1}{10}$$

(e) P (White and numbered higher than 12 or Yellow and numbered higher than 26)

= P (White and numbered higher than 12)

+ P (Yellow and numbered higher than 26)

$$= \frac{8}{80} + \frac{14}{80} = \frac{22}{80} = \frac{11}{40}$$

Objective Type Questions

Choose the correct answer from given four options in each of the Examples 9 to 15 (M.C.Q.).

Example 9 In a leap year the probability of having 53 Sundays or 53 Mondays is

(A) $\frac{2}{7}$ (B) $\frac{3}{7}$ (C) $\frac{4}{7}$ (D) $\frac{5}{7}$

Solution (B) is the correct answer. Since a leap year has 366 days and hence 52 weeks and 2 days. The 2 days can be SM, MT, TW, WTh, ThF, FSt, StS.

Therefore, P (53 Sundays or 53 Mondays) = $\frac{3}{7}$.

Example 10 Three digit numbers are formed using the digits 0, 2, 4, 6, 8. A number is chosen at random out of these numbers. What is the probability that this number has the same digits?

(A) $\frac{1}{16}$ (B) $\frac{16}{25}$ (C) $\frac{1}{645}$ (D) $\frac{1}{25}$

Solution (D) is the correct answer. Since a 3-digit number cannot start with digit 0, the hundredth place can have any of the 4 digits. Now, the tens and units place can have all the 5 digits. Therefore, the total possible 3-digit numbers are 4 × 5 × 5, i.e., 100.

The total possible 3 digit numbers having all digits same = 4

Hence, P (3-digit number with same digits) = $\frac{4}{100} = \frac{1}{25}$.

Example 11 Three squares of chess board are selected at random. The probability of getting 2 squares of one colour and other of a different colour is

(A) $\frac{16}{21}$ (B) $\frac{8}{21}$ (C) $\frac{3}{32}$ (D) $\frac{3}{8}$

Solution (A) is the correct answer. In a chess board, there are 64 squares of which 32 are white and 32 are black. Since 2 of one colour and 1 of other can be 2W, 1B, or 1W, 2B, the number of ways is $({}^{32}C_2 \times {}^{32}C_1) \times 2$ and also, the number of ways of choosing any 3 boxes is ${}^{64}C_3$.

Hence, the required probability = $\frac{{}^{32}C_2 \times {}^{32}C_1 \times 2}{{}^{64}C_3} = \frac{16}{21}$.

Example 12 If A and B are any two events having P $(A \cup B) = \frac{1}{2}$ and P $(\bar{A}) = \frac{2}{3}$, then the probability of $\bar{A} \cap B$ is

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$

Solution (C) is the correct answer. We have $P(A \cup B) = \frac{1}{2}$

$\Rightarrow$ P $(A \cup (B - A)) = \frac{1}{2}$

$\Rightarrow$ P (A) + P (B – A) = $\frac{1}{2}$ (since A and B – A are mutually exclusive)

$\Rightarrow$ $1 - P(\bar{A}) + P(B - A) = \frac{1}{2}$

$\Rightarrow$ $1 - \frac{2}{3} + P(B - A) = \frac{1}{2}$

$$\Rightarrow \quad P(B-A) = \frac{1}{6}$$

$$\Rightarrow \quad P(\bar{A} \cap B) = \frac{1}{6} \qquad \text{(since } \bar{A} \cap B \equiv B - A\text{)}$$

Example 13 Three of the six vertices of a regular hexagon are chosen at random. What is the probability that the triangle with these vertices is equilateral?

(A) $\frac{3}{10}$ (B) $\frac{3}{20}$ (C) $\frac{1}{20}$ (D) $\frac{1}{10}$

Solution (D) is the correct answer.

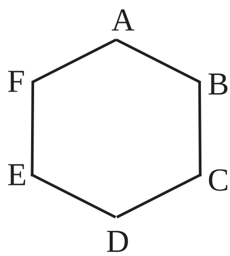


Fig. 16.1

ABCDEF is a regular hexagon. Total number of triangles ${}^6C_3 = 20$. (Since no three points are collinear). Of these only Δ ACE; Δ BDF are equilateral triangles.

Therefore, required probability $= \frac{2}{20} = \frac{1}{10}$.

Example 14 If A, B, C are three mutually exclusive and exhaustive events of an experiment such that

3P(A) = 2P(B) = P(C), then P(A) is equal to

(A) $\frac{1}{11}$ (B) $\frac{2}{11}$ (C) $\frac{5}{11}$ (D) $\frac{6}{11}$

Solution (B) is the correct answer. Let 3P (A) = 2P(B) = P(C) = p which gives p (A) $= \frac{p}{3}$, P(B) $= \frac{p}{2}$ and P(C) $= p$

Now since A, B, C are mutually exclusive and exhaustive events, we have

P(A) + P(B) + P(C) = 1

$$\Rightarrow \quad \frac{p}{3}+\frac{p}{2}+p=1 \qquad \Rightarrow \quad p=\frac{6}{11}$$

Hence, $P(A) = \frac{p}{3} = \frac{2}{11}$

Example 15 One mapping (function) is selected at random from all the mappings of the set A = {1, 2, 3, ..., n} into itself. The probability that the mapping selected is one to one is

(A) $\frac{1}{n^n}$ (B) $\frac{1}{\lfloor n}$ (C) $\frac{\lfloor n-1}{n^{n-1}}$ (D) none of these

Solution (C) is the correct answer. Total number of mappings from a set A having n elements onto itself is n^n

Now, for one to one mapping the first element in A can have any of the n images in A; the 2nd element in A can have any of the remaining (n – 1) images, counting like this, the n^{th} element in A can have only 1 image.

Therefore, the total number of one to one mappings is $\lfloor n$.

Hence the required probability is $\frac{\lfloor n}{n^n}=\frac{n\lfloor n-1}{n\,n^{n-1}} = \frac{\lfloor n-1}{n^{n-1}}$.

16.3 EXERCISE

Short Answer Type

1. If the letters of the word **ALGORITHM** are arranged at random in a row what is the probability the letters GOR must remain together as a unit?

2. Six new employees, two of whom are married to each other, are to be assigned six desks that are lined up in a row. If the assignment of employees to desks is made randomly, what is the probability that the married couple will have nonadjacent desks?

   [**Hint:** First find the probability that the couple has adjacent desks, and then subtract it from 1.]

3. Suppose an integer from 1 through 1000 is chosen at random, find the probability that the integer is a multiple of 2 or a multiple of 9.

4. An experiment consists of rolling a die until a 2 appears.

   (i) How many elements of the sample space correspond to the event that the 2 appears on the k^{th} roll of the die?

(ii) How many elements of the sample space correspond to the event that the 2 appears not later than the k^{th} roll of the die?

[**Hint:**(a) First $(k-1)$ rolls have 5 outcomes each and k^{th} rolls should result in 1 outcomes. (b)$1 + 5 + 5^2 + ... + 5^{k-1}$.]

5. A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find P(G), where G is the event that a number greater than 3 occurs on a single roll of the die.

6. In a large metropolitan area, the probabilities are .87, .36, .30 that a family (randomly chosen for a sample survey) owns a colour television set, a black and white television set, or both kinds of sets. What is the probability that a family owns either anyone or both kinds of sets?

7. If A and B are mutually exclusive events, P (A) = 0.35 and P (B) = 0.45, find

(a) $P(A')$ (b) $P(B')$ (c) $P(A \cup B)$ (d) $P(A \cap B)$

(e) $P(A \cap B')$ (f) $P(A' \cap B')$

8. A team of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rated as very complex, complex, routine, simple or very simple are respectively, 0.15, 0.20, 0.31, 0.26, .08. Find the probabilities that a particular surgery will be rated

(a) complex or very complex;

(b) neither very complex nor very simple;

(c) routine or complex

(d) routine or simple

9. Four candidates A, B, C, D have applied for the assignment to coach a school cricket team. If A is twice as likely to be selected as B, and B and C are given about the same chance of being selected, while C is twice as likely to be selected as D, what are the probabilities that

(a) C will be selected?

(b) A will not be selected?

10. One of the four persons John, Rita, Aslam or Gurpreet will be promoted next month. Consequently the sample space consists of four elementary outcomes

S = {John promoted, Rita promoted, Aslam promoted, Gurpreet promoted}

You are told that the chances of John's promotion is same as that of Gurpreet, Rita's chances of promotion are twice as likely as Johns. Aslam's chances are four times that of John.

(a) Determine P (John promoted)

P (Rita promoted)

P (Aslam promoted)

P (Gurpreet promoted)

(b) If A = {John promoted or Gurpreet promoted}, find P (A).

11. The accompanying Venn diagram shows three events, A, B, and C, and also the probabilities of the various intersections (for instance, $P(A \cap B) = .07$). Determine

(a) $P(A)$

(b) $P(B \cap \bar{C})$

(c) $P(A \cup B)$

(d) $P(A \cap \bar{B})$

(e) $P(B \cap C)$

(f) Probability of exactly one of the three occurs.

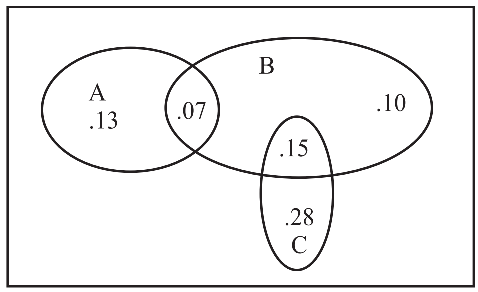


Fig. 16.2

Long Answer Type

12. One urn contains two black balls (labelled B1 and B2) and one white ball. A second urn contains one black ball and two white balls (labelled W1 and W2). Suppose the following experiment is performed. One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.

(a) Write the sample space showing all possible outcomes

(b) What is the probability that two black balls are chosen?

(c) What is the probability that two balls of opposite colour are chosen?

13. A bag contains 8 red and 5 white balls. Three balls are drawn at random. Find the Probability that

(a) All the three balls are white

(b) All the three balls are red

(c) One ball is red and two balls are white

14. If the letters of the word ASSASSINATION are arranged at random. Find the Probability that

(a) Four S's come consecutively in the word

(b) Two I's and two N's come together

(c) All A's are not coming together

(d) No two A's are coming together.

15. A card is drawn from a deck of 52 cards. Find the probability of getting a king or a heart or a red card.

16. A sample space consists of 9 elementary outcomes $e_1, e_2, ..., e_9$ whose probabilities are

$P(e_1) = P(e_2) = .08,\ P(e_3) = P(e_4) = P(e_5) = .1$

$P(e_6) = P(e_7) = .2,\ P(e_8) = P(e_9) = .07$

Suppose $A = \{e_1, e_5, e_8\}$, $B = \{e_2, e_5, e_8, e_9\}$

(a) Calculate P (A), P (B), and $P(A \cap B)$

(b) Using the addition law of probability, calculate $P(A \cup B)$

(c) List the composition of the event $A \cup B$, and calculate $P(A \cup B)$ by adding the probabilities of the elementary outcomes.

(d) Calculate $P(\overline{B})$ from P (B), also calculate $P(\overline{B})$ directly from the elementary outcomes of $\overline{B}$

17. Determine the probability *p*, for each of the following events.

(a) An odd number appears in a single toss of a fair die.

(b) At least one head appears in two tosses of a fair coin.

(c) A king, 9 of hearts, or 3 of spades appears in drawing a single card from a well shuffled ordinary deck of 52 cards.

(d) The sum of 6 appears in a single toss of a pair of fair dice.

Objective Type Questions

Choose the correct answer out of four given options in each of the Exercises 18 to 29 (M.C.Q.).

18. In a non-leap year, the probability of having 53 tuesdays or 53 wednesdays is

(A) $\frac{1}{7}$ (B) $\frac{2}{7}$ (C) $\frac{3}{7}$ (D) none of these

19. Three numbers are chosen from 1 to 20. Find the probability that they are not consecutive

(A) $\frac{186}{190}$ (B) $\frac{187}{190}$ (C) $\frac{188}{190}$ (D) $\frac{18}{{}^{20}C_3}$

20. While shuffling a pack of 52 playing cards, 2 are accidentally dropped. Find the probability that the missing cards to be of different colours

(A) $\frac{29}{52}$ (B) $\frac{1}{2}$ (C) $\frac{26}{51}$ (D) $\frac{27}{51}$

21. Seven persons are to be seated in a row. The probability that two particular persons sit next to each other is

(A) $\frac{1}{3}$ (B) $\frac{1}{6}$ (C) $\frac{2}{7}$ (D) $\frac{1}{2}$

22. Without repetition of the numbers, four digit numbers are formed with the numbers 0, 2, 3, 5. The probability of such a number divisible by 5 is

(A) $\frac{1}{5}$ (B) $\frac{4}{5}$ (C) $\frac{1}{30}$ (D) $\frac{5}{9}$

23. If A and B are mutually exclusive events, then

(A) $P(A) \le P(\bar{B})$ (B) $P(A) \ge P(\bar{B})$

(C) $P(A) < P(\bar{B})$ (D) none of these

24. If $P(A \cup B) = P(A \cap B)$ for any two events A and B, then

(A) $P(A) = P(B)$ (B) $P(A) > P(B)$

(C) $P(A) < P(B)$ (D) none of these

25. 6 boys and 6 girls sit in a row at random. The probability that all the girls sit together is

(A) $\frac{1}{432}$ (B) $\frac{12}{431}$ (C) $\frac{1}{132}$ (D) none of these

26. A single letter is selected at random from the word 'PROBABILITY'. The probability that it is a vowel is

(A) $\frac{1}{3}$ (B) $\frac{4}{11}$ (C) $\frac{2}{11}$ (D) $\frac{3}{11}$

27. If the probabilities for A to fail in an examination is 0.2 and that for B is 0.3, then the probability that either A or B fails is

(A) $> .5$ (B) $.5$ (C) $\le .5$ (D) 0

28. The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then $P(\bar{A}) + P(\bar{B})$ is

(A) 0.4 (B) 0.8 (C) 1.2 (D) 1.6

29. If M and N are any two events, the probability that at least one of them occurs is

(A) P (M) + P (N) – 2 P (M $\cap$ N) (B) P (M) + P (N) – P (M $\cap$ N)

(C) P (M) + P (N) + P (M $\cap$ N) (D) P (M) + P (N) + 2P (M $\cap$ N)

State whether the statements are True or False in each of the Exercises 30 to 36.

30. The probability that a person visiting a zoo will see the giraffee is 0.72, the probability that he will see the bears is 0.84 and the probability that he will see both is 0.52.

31. The probability that a student will pass his examination is 0.73, the probability of the student getting a compartment is 0.13, and the probability that the student will either pass or get compartment is 0.96.

32. The probabilities that a typist will make 0, 1, 2, 3, 4, 5 or more mistakes in typing a report are, respectively, 0.12, 0.25, 0.36, 0.14, 0.08, 0.11.

33. If A and B are two candidates seeking admission in an engineering College. The probability that A is selected is .5 and the probability that both A and B are selected is at most .3. Is it possible that the probability of B getting selected is 0.7?

34. The probability of intersection of two events A and B is always less than or equal to those favourable to the event A.

35. The probability of an occurrence of event A is .7 and that of the occurrence of event B is .3 and the probability of occurrence of both is .4.

36. The sum of probabilities of two students getting distinction in their final examinations is 1.2.

Fill in the blanks in the Exercises 37 to 41.

37. The probability that the home team will win an upcoming football game is 0.77, the probability that it will tie the game is 0.08, and the probability that it will lose the game is _____.

38. If e_1, e_2, e_3, e_4 are the four elementary outcomes in a sample space and P(e_1) = .1, P(e_2) = .5, P (e_3) = .1, then the probability of e_4 is ______.

39. Let S = {1, 2, 3, 4, 5, 6} and E = {1, 3, 5}, then $\overline{\text{E}}$ is _________.

40. If A and B are two events associated with a random experiment such that P (A) = 0.3, P (B) = 0.2 and P (A $\cap$ B) = 0.1, then the value of P (A $\cap$ $\overline{\text{B}}$) is ________.

41. The probability of happening of an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive events, then the probability of neither A nor B is ________.

42. Match the proposed probability under Column C_1 with the appropriate written description under column C_2 :

C_1 Probability	C_2 Written Description
(a) 0.95	(i) An incorrect assignment
(b) 0.02	(ii) No chance of happening
(c) – 0.3	(iii) As much chance of happening as not.
(d) 0.5	(iv) Very likely to happen
(e) 0	(v) Very little chance of happening

43. Match the following

(a) If E_1 and E_2 are the two mutually exclusive events	(i) $E_1 \cap E_2 = E_1$
(b) If E_1 and E_2 are the mutually exclusive and exhaustive events	(ii) $(E_1 - E_2) \cup (E_1 \cap E_2) = E_1$
(c) If E_1 and E_2 have common outcomes, then	(iii) $E_1 \cap E_2 = \phi$, $E_1 \cup E_2 = S$
(d) If E_1 and E_2 are two events such that $E_1 \subset E_2$	(iv) $E_1 \cap E_2 = \phi$